Announcements

- **Topics:** Concavity, asymptotes, graphing
- **Homework:** Watch videos 7.1 - 7.6 before the start of next semester!

Computations

Calculate:

1. $\lim_{x\to 2} \dfrac{x^2+2x-6}{x^2+3x-10}$
2. $\lim_{x\to\infty} x^3 e^{-x}$
3. $\lim_{x\to 0} x \sin \dfrac{2}{x}$
4. $\lim_{x\to\infty} x \sin \dfrac{2}{x}$
5. $\lim_{x\to\infty} x \cos \dfrac{2}{x}$
6. $\lim_{x\to 0} \dfrac{e^{2x^2} - \cos x}{x \sin x}$
7. $\lim_{x\to\infty} \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$
8. $\lim_{x\to 1} \left[(\ln x) \tan \dfrac{\pi x}{2} \right]$

Infinity minus infinity

Calculate:

1. $\lim_{x \to 0} \left[\frac{\csc x}{x} - \frac{\cot x}{x} \right]$
2. $\lim_{x \to \infty} [\ln(x+2) - \ln(3x+4)]$
3. $\lim_{x \to 1} \left[\frac{2}{x^2-1} - \frac{1}{x-1} \right]$
4. $\lim_{x \to -\infty} \left[\sqrt{x^2+3x} - \sqrt{x^2-3x} \right]$

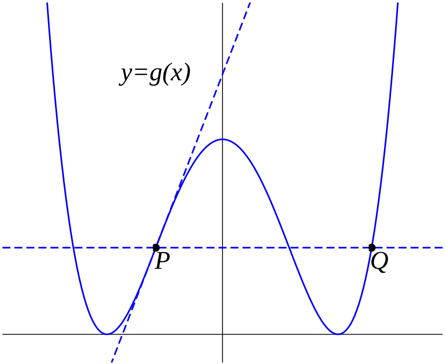
Exponential indeterminate forms

Calculate:

1. $\lim_{x\to\infty}\left(\frac{x+2}{x-2}\right)^{3x}$
2. $\lim_{x\to 0}\,[1+2\sin(3x)]^{4\cot(5x)}$
3. $\lim_{x\to 0^+} x^x$
4. $\lim_{x\to \frac{\pi}{2}^-} (\tan x)^{\cos x}$
5. $\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{1/x^2}$

Find the coordinates of P and Q

$$g(x) = x^4 - 6x^2 + 9$$



Unusual examples

Construct a function f such that

- the domain of f is at least $(0, \infty)$
- f is continuous and concave up on its domain
- $\lim_{x \to \infty} f(x) = -\infty$

Construct a function g such that

- the domain of g is $\mathbb{R}$
- g is continuous
- g has a local minimum $x = 0$
- g has an inflection point at $x = 0$

Asymptotics

$f(x)$ asymptotic to $g(x)$ as $x \to \infty$

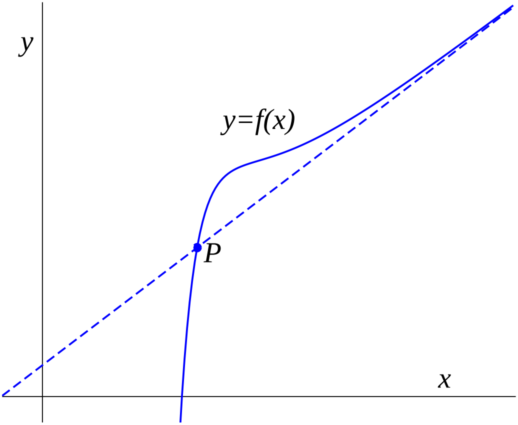
We say $f(x)$ is asymptotic to $g(x)$ as $x \to \infty$ if $\lim_{x\to\infty} [f(x) - g(x)] = 0$

There's a similar definition for $f(x)$ asymptotic to $g(x)$ as $x \to -\infty$.

Often we will be interested in if $f(x)$ is asymptotic to a line $L(x)$.

Find the coordinates of P

$$f(x) = 3x + 4 + \frac{2x - 10}{x^2}$$



Asymptotics

Find the line asymptotes of $f(x) = \sqrt{x^2 + 4x}$ as $x \to \infty$ and $x \to -\infty$.

Hint: Assume $\exists a, b \in \mathbb{R}$ s.t. $ax + b$ is asymptotic to $f(x)$ as $x \to \infty$. Write down what this means and then compute what a and b has to be.

Line asymptotes

Let f be defined on $\mathbb{R}$.

We want to see if there are $a, b \in \mathbb{R}$ such that $\lim_{x\to\infty} [f(x) - ax - b] = 0$.

What does a have to be? HINT: Suppose $\exists a, b \in \mathbb{R}$ s.t. the limit equation is satisfied, divide by x and solve for a.

Now we have a formula for a, come up with a way to find b using the previous a.

Line asymptotes

Theorem

If $\lim_{x \to \infty} \frac{f(x)}{x}$ exists and equal a, and $\lim_{x \to \infty}[f(x) - ax]$ exists and equal b, then $f(x)$ is asymptotic to the line $L(x) = ax + b$ as $x \to \infty$. If any of the two limits DNE, then there are no lines asymptotic to f as $x \to \infty$

Theorem

If $\lim_{x \to -\infty} \frac{f(x)}{x}$ exists and equal a, and $\lim_{x \to -\infty}[f(x) - ax]$ exists and equal b, then $f(x)$ is asymptotic to the line $L(x) = ax + b$ as $x \to -\infty$. If any of the two limits DNE, then there are no lines asymptotic to f as $x \to -\infty$

Horizontal and Slant asymptotes

In the case line asymptotes exist as $x \to \pm\infty$, we have two cases:

When $a = 0$, we call the line asymptote a horizontal asymptote. In other words,

Horizontal asymptote

We say f has a horizontal asymptote of b_1 as $x \to \infty$ iff $\lim_{x\to\infty} [f(x) - 0x] = b_1$ iff $\lim_{x\to\infty} f(x) = b_1$

We say f has a horizontal asymptote of b_2 as $x \to -\infty$ iff $\lim_{x\to-\infty} [f(x) - 0x] = b_2$ iff $\lim_{x\to-\infty} f(x) = b_2$

When $a \neq 0$, we call the line asymptote a slant asymptote. In particular, as $x \to \infty$ or as $x \to -\infty$, slant and horizontal asymptotes are mutually exclusive.

Vertical asymptotes vs vertical tangencies

Let f be a function.

1. Write the definition of "f has a vertical asympote at $x = 1$".
2. Write the definition of "f has a vertical tangent line at $x = 1$".

1. Graph a function g which has a vertical asymptote at $x = 1$.
2. Graph a function h which has a veritcal tangent line at $x = 1$.

Periodic?

Construct a function H that has all the following properties at once:

1. The domain of H is $\mathbb{R}$
2. H is strictly increasing on $\mathbb{R}$
3. H is differentiable on $\mathbb{R}$
4. H' is periodic with a period of 2
5. H' is not constant

A weird function

The function $G(x) = xe^{1/x}$ is deceiving. To help you out:

$$G'(x) = \frac{x-1}{x}e^{1/x}, \qquad G''(x) = \frac{e^{1/x}}{x^3}$$

1. Carefully study the behaviour as $x \to \pm\infty$.
   You should find line asymptotes, but it is not easy.
2. Carefully study the behaviour as $x \to 0^+$ and $x \to 0^-$.
   The two are very different.
3. Use G' to study monotonocity.
4. Use G'' to study concavity.
5. Sketch the graph of G.