Announcements

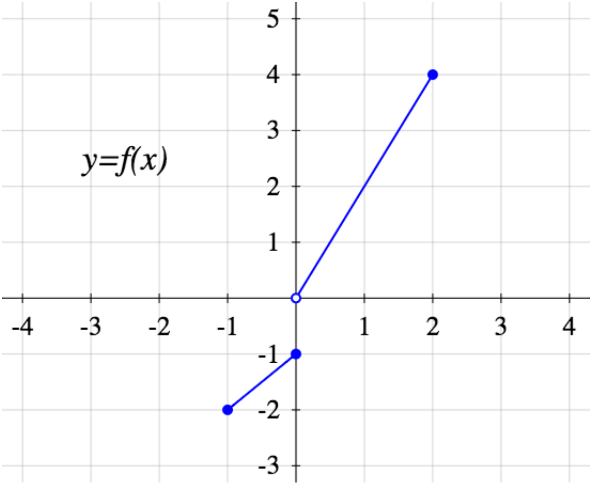
- **Topics:** Inverse functions, one-to-one functions, inverse trig functions, local extrema
- **Homework:** Watch videos 5.5 - 5.12

An interesting example

Let $f(x) = x^2 \sin \frac{1}{x}$.

1. Calculate $f'(x)$ for any $x \neq 0$.
2. Using the definition of derivative, calculate $f'(0)$.
3. Is f continuous at 0?
4. Is f differentiable at 0?
5. Is f' continuous at 0?

Inverse function from a graph



Calculate:

1. $f(2)$
2. $f(0)$
3. $f^{-1}(2)$
4. $f^{-1}(0)$
5. $f^{-1}(-1)$

Absolute value and inverses

Let

$$h(x) = x|x| + 1$$

1. Calculate $h^{-1}(-8)$.
2. Find an equation for $h^{-1}(x)$.
3. Sketch the graphs of h and h^{-1}.

Composition and inverses

Assume for simplicity that all functions in this problem have domain $\mathbb{R}$.

Let f and g be functions. Assume they each have an inverse.

Is $(f \circ g)^{-1} = f^{-1} \circ g^{-1}$?

- If YES, prove it.
- If NO, fix the statement.

If you do not know how to start, experiment with the functions

$$f(x) = x + 1, \qquad g(x) = 2x.$$

Composition of one-to-one functions – 2

Assume for simplicity that all functions in this problem have domain $\mathbb{R}$.

Is the following claim TRUE or FALSE? Prove it or give a counterexample.

Claim

Let f and g be functions.
IF $f \circ g$ is one-to-one,
THEN g is one-to-one.

Homework: Composition of one-to-one functions – 3

Assume for simplicity that all functions in this problem have domain $\mathbb{R}$.

Is the following claim TRUE or FALSE? Prove it or give a counterexample.

Claim

Let f and g be functions.
IF $f \circ g$ is one-to-one,
THEN f is one-to-one.

Derivative of the inverse

Let f be one-to-one.

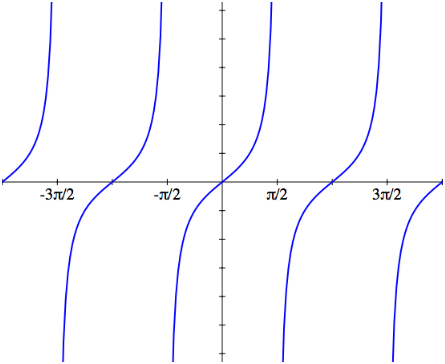
Let $a, b \in \mathbb{R}$ s.t. $f(a) = b$.

Suppose both f and f^{-1} are twice differentiable.

1. Find a formula for $(f^{-1})'(b)$ involving $f'(a)$.
2. Find a formula for $(f^{-1})''(b)$ involving $f'(a)$ and $f''(a)$.

The arctan function

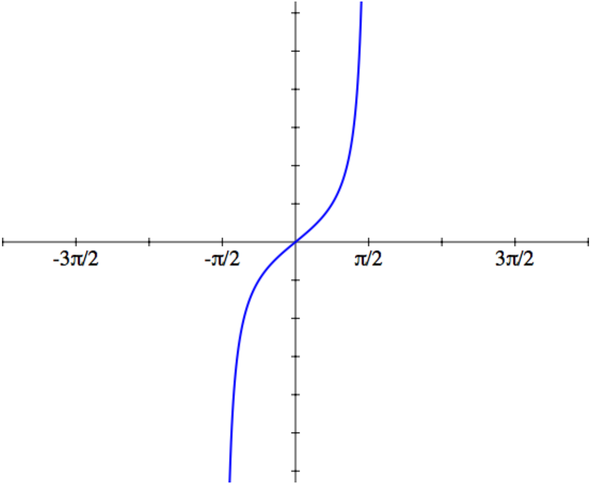
Here's (part of) the graph of the tan function.



Question. Does this function have an inverse?
Problem. Find the largest interval containing 0 such that the restriction of tan to it is injective.

The arctan function

We define arctan to be the inverse of the function with this graph:



The arctan function

In symbols, that means we define the function arctan as the inverse of the function

$$g(x) = \tan x, \text{ restricted to the interval } \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

In other words, if $x, y \in \mathbb{R}$, then

$$\arctan(y) = x \quad \iff \quad \begin{cases} ??? \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \\ ??? \end{cases}$$

Problem 1. What should be where the question marks are?

Problem 2. What are the domain and range of arctan?

Problem 3. Sketch the graph of arctan.

The arctan function

To remind you:

$$\arctan(y) = x \iff \begin{cases} x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \\ \tan x = y \end{cases}$$

Compute the following values:

1. $\arctan\left(\tan\left(1\right)\right)$
2. $\arctan\left(\tan\left(3\right)\right)$
3. $\arctan\left(\tan\left(\frac{\pi}{2}\right)\right)$
4. $\arctan\left(\tan\left(-6\right)\right)$
5. $\tan\left(\arctan\left(0\right)\right)$
6. $\tan\left(\arctan\left(10\right)\right)$

Standard choice of restrictions

We make the following standard choice of restrictions when we define the inverse trig functions:

1. $\sin(x)$ restricted to $[-\frac{\pi}{2}, \frac{\pi}{2}]$.
2. $\cos(x)$ restricted to $[0, \pi]$.
3. $\tan(x)$ restricted to $(-\frac{\pi}{2}, \frac{\pi}{2})$.
4. $\sec(x)$ restricted to $[0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$.
5. $\csc(x)$ restricted to $[-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2}]$.
6. $\cot(x)$ restricted to $(0, \pi)$.

Developing $\arctan_2$

Let's define $\arctan_2(x)$ as the inverse of the restriction of $\tan(x)$ to the interval $(\frac{\pi}{2}, \frac{3\pi}{2})$. Find the following:

1. The domain and the range of $\arctan_2$.

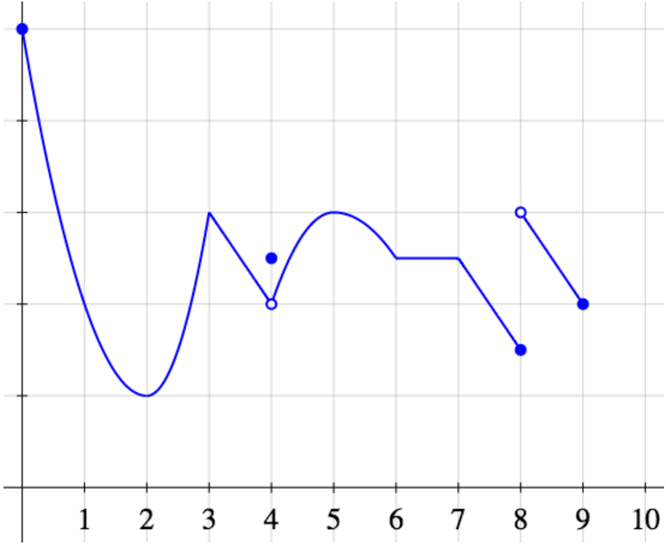
2. A graph of $\arctan_2$.

3. $\tan(\arctan_2(12))$, $\arctan_2(\tan(0))$, $\arctan_2(\tan(\pi))$, $\arctan_2(\tan(7))$

4. Compute the derivative of $\arctan_2$.

Definition of local extremum

Find local and global extrema of the function with this graph:



Where is the local extrema?

We know the following about the function h:

- The domain of h is $(-4, 4)$.
- h is continuous on its domain.
- h is differentiable on its domain, except at 0.
- $h'(x) = 0 \iff x = -1$ or 1.

What can you conclude about the local extrema of h?

1. h has a local extrema at $x = -1$, or 1.
2. h has a local extrema at $x = -1, 0$, or 1.
3. h has a local extrema at $x = -4, 1, 0, 1$, or 4.
4. None of the above.

Fractional exponents

Let $g(x) = x^{2/3}(x-1)^3$.

Find local and global extrema of g on $[-1, 2]$.

Let $h(x) = x^4 - 4x$.

Find local and global extrema of h on $\mathbb{R}$.

A sneaky function

Construct a function f satisfying all the following properties:

- Domain $f = \mathbb{R}$
- f is continuous
- $f'(0) = 0$
- f does not have a local extremum at 0.
- There isn't an interval centered at 0 on which f is increasing.
- There isn't an interval centered at 0 on which f is decreasing.

What can you conclude?

We know the following about the function f.

- f has domain $\mathbb{R}$.
- f is continuous
- $f(0) = 0$
- For every $x \in \mathbb{R}$, $f(x) \geq x$.

What can you conclude about $f'(0)$? Prove it.

Hint: Sketch the graph of f. Looking at the graph, make a conjecture.
To prove it, imitate the proof of the Local EVT from Video 5.3.