Announcements

- **Topics:** Proofs of differentiation rules, Chain rule, trig derivatives, implicit differentiation
- **Homework:** Watch videos 3.13 - 3.20, 4.1, 4.2

Product of 3 functions

Given f_1, f_2 and f_3 differentiable on $\mathbb{R}$, what can you say $(f_1(x)f_2(x)f_3(x))' =$?

Higher order derivatives

Let $g(x) = \dfrac{1}{x^3}$.

Calculate the first few derivatives.
Make a conjecture for a formula for the n-th derivative $g^{(n)}(x)$.
Prove it.

Homework: Computations

Compute the derivative of the following functions:

1. $$f(x) = x^{100} + 3x^{30} - 2x^{15}$$
2. $f(x) = \sqrt[3]{x} + 6$
3. $f(x) = \dfrac{4}{x^4}$
4. $f(x) = \sqrt{x}\,(1 + 2x)$
5. $f(x) = \dfrac{x^6 + 1}{x^3}$
6. $f(x) = \dfrac{x^2 - 2}{x^2 + 2}$

Warm-up: differentiability implies continuity

Let $a \in \mathbb{R}$.

Let f be defined in a neighbourhood of a.

Write the definitions of "f is continuous at a" and "f is differentiable at a" using limits.

1. Prove if f is differentiable at a then f is continuous a.
2. Show it's not necessarily true that if f is continuous a then f is differentiable a.

A lemma

Let $a \in \mathbb{R}$.

Given a function f defined in a neighbourhood of a.

Assume f is continuous at a and $f(a) \neq 0$.

Prove $\exists \delta > 0$ s.t. $\forall x \in (a - \delta, a + \delta)$, $f(x) \neq 0$.

Quotient rule

Let $a \in \mathbb{R}$.

Given functions f and g defined in a neighbourhood of a.

Define $h(x) = \frac{f(x)}{g(x)}$.

Assume f and g are ________________.

Assume __________________________.

Then ____.

Prove this.

Things to check

1. Did you start with the definition of derivative for h at a?

2. Did you check $h(x) = \frac{f(x)}{g(x)}$ is actually defined in a neighbourhood of a. (Is it necessary to check this?)

3. Did you assume at some point a function is differentiable? If so, did you justify it?

4. Did you assume at some point a function is continuous? If so, did you justify it? (This has to come up in the proof somewhere.)

5. Does the proof work?

6. Can a competent reader understand your proof without asking you for clarification?

Warm-up

Compute the derivative of the following (do not worry too much about the domain):

1. $f(x) = \sqrt{2x^2 + x + 1}$
2. $g(x) = \sqrt{x + \sqrt{x + \sqrt{x + 1}}}$

Two different limits

Let f be differentiable on $\mathbb{R}$. Compute the following limits in terms of $f'(x)$.

1. $\displaystyle\lim_{h\to 0} \frac{f(2x+h)-f(2x)}{h}$
2. $\displaystyle\lim_{h\to 0} \frac{f(2(x+h))-f(2x)}{h}$

Higher order derivatives of $f \circ g$

Assume f and g have derivatives of all order.

Find formulas for:

1. $(f \circ g)'(x)$
2. $(f \circ g)''(x)$
3. $(f \circ g)'''(x)$

in terms of the values of f, g and their derivatives of any order.

Another “proof” of the quotient rule

Use the product rule, the chain rule and the power rule to prove the quotient rule.

Hint: Start by writing the quotient using products, powers and compositions instead.

Compute the derivative of $\cos(x)$ from the definition of derivative as a limit.

Hint: Write down the limit and try to imitate what was done for $\sin(x)$ in the videos. You will need a trig identity.

Homework: Derivative of the other trig functions

Using the differentiation rules and

$$\frac{d}{dx}\sin(x) = \cos(x), \qquad \frac{d}{dx}\cos(x) = -\sin(x).$$

Find:

1. $\frac{d}{dx}\tan(x)$
2. $\frac{d}{dx}\cot(x)$
3. $\frac{d}{dx}\sec(x)$
4. $\frac{d}{dx}\csc(x)$

Implicit Differentiation

Function: For each input there is a unique output

Relation: A relationship between several variables with no well-defined idea of an input and an output, in particular no “uniqueness” of output.

Example: $x^2 + y^2 = 1$ is a relation but not a function.

We can still graph the relation by drawing the curve(s) of all (x, y) satisfying the equation and talk about the “tangent slope” at a given point on the graph.

However, saying something like find $\left.\frac{dy}{dx}\right|_{x=0}$ (usually) doesn't make sense. Why?

Implicit differentiation

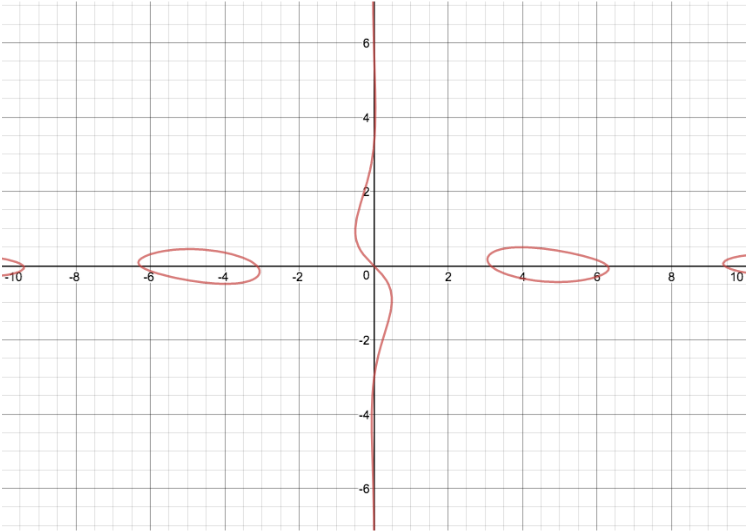
The equation

$$\sin(x + y) + xy^2 = 0$$

defines a function $y = h(x)$ near $(0, 0)$.

Compute:

1. $h(0)$
2. $h'(0) = \left.\frac{dy}{dx}\right|_{x=0,y=0}$
3. $h''(0) = \left.\frac{d^2y}{dx^2}\right|_{x=0,y=0}$
4. $h'''(0) = \left.\frac{d^3y}{dx^3}\right|_{x=0,y=0}$


6
4
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0
-2
-4
-6
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-8
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2
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8
10

Implicit differentiation

1. What is $\left.\frac{dx}{dy}\right|_{x=0,y=0}$? Make a guess from your previous work and check it by implicit differentiation.

2. What is $\left.\frac{d^2x}{dy^2}\right|_{x=0,y=0}$?