Announcements

- **Topics:** Continuity, lots of limit proofs, behaviour of limits under composition
- **Homework:** Watch videos 2.21, 2.22, 3.1 - 3.5, 3.8

Another squeeze theorem

Another squeeze theorem

Let $a \in \mathbb{R}$. Let f and g be functions defined near a, except possibly at a.

IF

- $\exists p > 0$ s.t. $0 < |x - a| < p \implies f(x) \geq g(x)$.
- $\lim_{x \to a} g(x) = \infty$.

THEN

- $\lim_{x \to a} f(x) = \infty$.

Prove this theorem.

Hint: The proof of this theorem is similar to but easier than the standard squeeze theorem. Write down the relevant $M - \delta$ definitions and try to prove one from the other.

Product limit law

Product limit law

Let $a \in \mathbb{R}$.
Let f and g be functions defined near a, except possibly at a.
IF $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist
THEN $\lim_{x \to a} f(x)g(x)$ exist and $= (\lim_{x \to a} f(x))(\lim_{x \to a} g(x))$.

1. Write down some formal definitions.
2. Choose L and M to be the limits of f and g at a respectively.
3. Since you can control $|x - a|$, what 2 other quantities can you control from your assumption? What are you trying to control?
4. Start with what you are trying to control. Algebraically manipulate it to produce an expression contaning the quantities you can control. (Hint: add and subtract by the same quantity inside the absolute value.)

Continuity

For a function f defined on an open interval of a, we say $f(x)$ is cts at a iff

Definition 1

$$\lim_{x \to a} f(x) = f(a)$$

This is clearly equivalently to

Definition 2

$\forall \epsilon > 0, \exists \delta > 0$ s.t. $0 < |x - a| < \delta \implies |f(x) - f(a)| < \epsilon$.

Slightly less clearly, it is also equivalent to

Definition 3

$\forall \epsilon > 0, \exists \delta > 0$ s.t. $|x - a| < \delta \implies |f(x) - f(a)| < \epsilon$.

Continuity on different sets

Continuous at a point

f continuous at c means $\lim_{x \to c} f(x) = f(c)$.

Continuous on an open interval

f continuous on the interval (a, b) means $\forall c \in (a, b)$, f is continuous at c.

Continuous on a closed interval

f continuous on the interval $[a, b]$ means

1. $\lim\limits_{x \to a^+} f(x) = f(a)$
2. $\forall c \in (a, b)$, f is continuous at c
3. $\lim\limits_{x \to b^-} f(x) = f(b)$

Dirichlet function

Consider the Dirichlet function

$$D(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R} \backslash \mathbb{Q} \end{cases}$$

1. Write the definition of $\lim_{x \to 0} D(x) \neq 0.5$.
2. Prove it.
3. Write the definition of $\lim_{x \to 0} D(x)$ DNE.
4. Exercise: Prove 3.

Continuity examples

Find examples of a function defined on $\mathbb{R}$ satisfying the following conditions:

1. $f(x)$ is continuous on $\mathbb{R}$.
2. $g(x)$ is continuous at every $c \in \mathbb{R} \setminus \{0\}$ and discontinuous at 0.
3. $h(x)$ is discontinuous at every $c \in \mathbb{R}$.
4. $m(x)$ is continuous at 0 and discontinuous at every $c \in \mathbb{R}$.

Hint: Try adjusting the Dirichlet function.

Behaviour of limits under composition

From examples we've seen before, in general, it is not true that

$$\lim_{x \to a} f(g(x)) = f(\lim_{x \to a} g(x)).$$

However, the statement does become true if f is continuous (more specifically, at $\lim_{x \to a} g(x)$).

Theorem: limit "commutes" with continuous functions

IF $\lim_{x \to a} g(x)$ exists and f is continuous at $\lim_{x \to a} g(x)$.
THEN $\lim_{x \to a} f(g(x)) = f(\lim_{x \to a} g(x))$.

Behaviour of limits under composition

Prove the theorem (assume for simplicity that f and g are defined on $\mathbb{R}$).

Theorem: limit "commutes" with continuous functions

IF $\lim_{x \to a} g(x)$ exists and f is continuous at $\lim_{x \to a} g(x)$.
THEN $\lim_{x \to a} f(g(x)) = f(\lim_{x \to a} g(x))$.

1. For simplicity of writing let L be $\lim_{x \to a} g(x)$. Write down your two assumptions in $\epsilon - \delta$ form.
2. Write down what you are trying to prove in $\epsilon - \delta$ form.
3. Prove it. Hint: Think about what the two assumptions give you control over. You are going to have to use the δ you get from one of your assumptions as the ϵ in the other assumption.

Behaviour of limits under composition

Fill in the blank and then prove the claim.

Claim

Let $a, L \in \mathbb{R}$.

Let f be a function defined on a punctured neighbourhood of a (i.e. on some open neighbourhood of a, except possibly at a).

If $\lim_{x \to a} f(x) = L$

Then $\lim_{x \to \frac{a}{5}} 2f(5x) =$ _____.

Notice this theorem is not a consequence of the previous theorem becaues f is not assumed to be continuous.

Computations

Suppose $\lim_{x \to a} f(x) = L$, then $\lim_{x \to \frac{a}{k}} f(kx) = L$.

Compute:

1. $\lim_{x \to 0} \frac{\sin(3x)}{x}$
2. $\lim_{x \to 0} \frac{1-\cos(x)}{x}$

Computations using limit laws

Given a function g s.t.

$$\lim_{x \to 0} \frac{g(x)}{x^2} = 2.$$

Use it to compute the following limits (or explain that they don't exist).

1. $\lim_{x \to 0} \frac{g(x)}{x}$
2. $\lim_{x \to 0} \frac{g(x)}{x^4}$
3. $\lim_{x \to 0} \frac{g(3x)}{x^2}$

Homework: Computations

Compute:

1. $\lim_{x\to 2} \frac{|x^2-4|}{x^2-5x+6}$
2. $\lim_{x\to 4} \frac{x^2-5x+4}{\sqrt{x}-2}$
3. $\lim_{x\to \infty} \frac{x^3+\sqrt{2x^6+1}}{2x^3+\sqrt{x^5+1}}$
4. $\lim_{x\to -\infty} x - \sqrt{x^2+x}$
5. $\lim_{x\to -\infty} x + \sqrt{x^2+x}$