Announcements

- **Topics:** Formal proofs with limit, limit laws, and the Squeeze Theorem
- **Homework:** Watch videos 2.14 - 2.20

Lmits involving infinity

Given $a, L \in \mathbb{R}$.

Write down the definition of the following statments:

1. $\lim_{x \to a} f(x) = \infty$.
2. $\lim_{x \to \infty} f(x) = L$.

Hint: For 1, you want to replace the two parts with ϵ in the $\epsilon - \delta$ definition for limits. Instead of saying $f(x)$ gets arbitrarily close to L as x gets close to a, you want to say $f(x)$ gets arbitrarily large. How can you do this?

Warm-up

Let $x \in \mathbb{R}$ and S_1, S_2, S_3 and S_4 be logic statements. Suppose you know:

1. If $|x-2|<4$, then S_1 (is true).
2. If $|x-2|<5$, then S_2 (is true).

What condition do you need to guarantee S_1 and S_2 are both true?

Suppose you know:

1. If $x>100$, then S_3 (is true).
2. If $x>1000$, then S_4 (is true).

What condition do you need to guarantee S_3 and S_4 are both true?

Warm-up

1. Find a value of $\delta > 0$ s.t.

$$|x-2| < \delta \implies |2x-4| < 1.$$

2. Find all values of $\delta > 0$ s.t.

$$|x-2| < \delta \implies |2x-4| < 1.$$

3. Find all values of $\delta > 0$ s.t.

$$|x-2| < \delta \implies |2x-4| < 0.1.$$

3. Let $\epsilon > 0$, find all values of $\delta > 0$ s.t.

$$|x-2| < \delta \implies |2x-4| < \epsilon.$$

An $\epsilon - \delta$ proof

Goal

Prove

$$\lim_{x \to 2} 2x = 4$$

from the definition.

1. Write down the formal definition of claim. This is the statement you will need to prove.
2. Write down the structure of the proof without details.
3. Write down the complete proof.

An $\epsilon - \delta$ proof

Goal

Prove
$$\lim_{x \to 2} x^2 = 4$$
from the definition.

1. Write down the formal definition of claim.

2. Write down the structure of the proof without details. **Just do the first 2 step for now.**

3. Write down the complete proof.

Is this proof correct?

Claim

$\lim_{x \to 2} x^2 = 4$

Proof:

Let $\epsilon > 0$.
Choose $\delta = \frac{\epsilon}{|x+2|}$.
Let $x \in \mathbb{R}$.
Asssume $0 < |x - 2| < \delta$, then,

$$|x^2 - 4| = |x - 2||x + 2| < \frac{\epsilon}{|x + 2|}|x + 2| = \epsilon.$$

$\square$

Steps for doing an $\epsilon - \delta$ proof rough work

Goal

Prove

$$\lim_{x \to 2} x^2 = 4$$

from the definition.

1. Write down the formal definition of claim. What can you control? What are you trying to control?

2. Start with the $|f(x) - L|$ part of the defintion. Algebraically manipulate it to get several terms.

3. Determine which one of the terms you can make arbitrarily small by constraining $|x - a|$.

4. Bound all other terms by constants by constraining $|x - a|$.

An $\epsilon - \delta$ proof

Goal

Prove

$$\lim_{x \to 2} x^2 = 4$$

from the definition.

1. Write down the formal definition of claim.
2. Write down the structure of the proof without details.
3. Write down the complete proof.

Another $\epsilon - \delta$ proof

Goal

Prove

$$\lim_{x \to 4} \frac{3}{x} = \frac{3}{4}$$

from the definition.

1. Write down the formal definition of claim. What can you control? What are you trying to control?

2. Start with the $|f(x) - L|$ part of the defintion. Algebraically manipulate it to get several terms.

3. Determine which one of the terms you can make arbitrarily small by constraining $|x - a|$.

4. Bound all other terms by constants by constraining $|x - a|$.

5. Write down a formal proof.

True or False?

True or false?

Claim

Let $a \in \mathbb{R}$.

Let f and g be functions defined near a.

- IF $\lim_{x \to a} f(x) = 0$,
- THEN $\lim_{x \to a} [f(x)g(x)] = 0$.

Indeterminate form

Let $a \in \mathbb{R}$.
Let f and g be functions defined near a.
Assume $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$.

What can we conclude about $\displaystyle \lim_{x \to a} \frac{f(x)}{g(x)}$?

1. The limit is 1 the 0s cancel.
2. The limit does not exist because it's $\frac{0}{0}$.
3. We do not have enough information to decide.

Theorem

Definition

Given a function f defined on some domain $D \subseteq \mathbb{R}$.
We say f is bounded iff $\exists M \in \mathbb{R}$ s.t. $\forall x \in D$, $|f(x)| < M$.

Prove the following claim:

Claim

Let $a \in \mathbb{R}$.
Let f and g be functions defined near a. Let g be bounded.

- IF $\lim_{x \to a} f(x) = 0$,
- THEN $\lim_{x \to a} [f(x)g(x)] = 0$.

Hint: Bound $f(x)g(x)$ by appropriate functions, use the squeeze theorem.

Another squeeze theorem

Another squeeze theorem

Let $a \in \mathbb{R}$. Let f and g be functions defined near a, except possibly at a.

IF

- For x close to a but not a, $f(x) \geq g(x)$.
- $\lim_{x \to a} g(x) = \infty$.

THEN

- $\lim_{x \to a} f(x) = \infty$.

Another squeeze theorem

Another squeeze theorem

Let $a \in \mathbb{R}$. Let f and g be functions defined near a, except possibly at a.

IF

- $\exists p > 0$ s.t. $0 < |x - a| < p \implies f(x) \geq g(x)$.
- $\lim_{x \to a} g(x) = \infty$.

THEN

- $\lim_{x \to a} f(x) = \infty$.

Prove this theorem.

Hint: The proof of this theorem is similar to but easier than the standard squeeze theorem. Write down the relevant $M - \delta$ definitions and try to prove one from the other.

Product limit law

Product limit law

Let $a \in \mathbb{R}$.
Let f and g be functions defined near a, except possibly at a.
IF $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist
THEN $\lim_{x \to a} f(x)g(x)$ exist and $= (\lim_{x \to a} f(x))(\lim_{x \to a} g(x))$.

1. Write down some formal definitions.
2. Choose L and M to be the limits of f and g at a respectively. Why can you do this?
3. Since you can control $|x - a|$, what 2 other quantities can you control from your assumption? What are you trying to control?
4. Start with what you are trying to control. Algebraically manipulate it to see the quantities you can control. (Hint: add and subtract by the same quantity inside the absolute value.)