Announcements

- **Topics:** Concavity, Asymptotes, Curve sketching

Fire

You hear a scream. You turn around and you see that Qin is on fire. Literally.

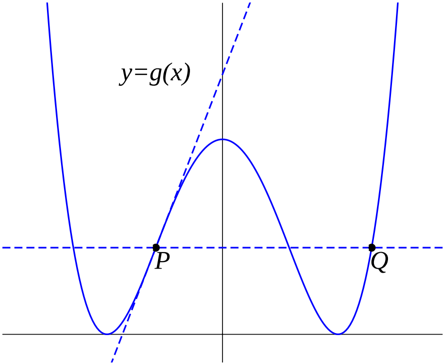
At first, you think maybe you should just let Qin burn - perhaps they'll cancel the test if that happens. After a moment, your pesky conscience sets in.

Luckily, you are next to a river.

Qin is 10 meters away from the river and you are 5 meters away from the point P on the river closest to Qin. You are carrying an empty bucket. You can run twice as fast with an empty bucket as you can run with a full bucket. How far from the point P should you fill your bucket in order to get to Qin with a bucket full of water as fast as possible?

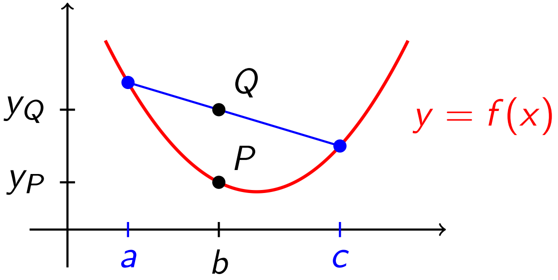
Find the coordinates of P and Q

$$g(x) = x^4 - 6x^2 + 9$$



"Secant segments are above the graph"

Let f be a function defined on an interval I.
In Video 6.11 you learned that an alternative way to define "f is concave up on I" is to say that "the secant segments stay above the graph".



Rewrite this as a precise mathematical statement of the form

$$\text{"}\forall a, b, c \in I, \quad a < b < c \implies \boxed{\text{an inequality involving } f,\ a,\ b,\ c}\text{"}$$

Unusual examples

Construct a function f such that

- the domain of f is at least $(0, \infty)$
- f is continuous and concave up on its domain
- $\lim_{x\to\infty} f(x) = -\infty$

Construct a function g such that

- the domain of g is $\mathbb{R}$
- g is continuous
- g has a local minimum $x = 0$
- g has an inflection point at $x = 0$

Asymptotics

$f(x)$ asymptotic to $g(x)$ as $x \to \infty$

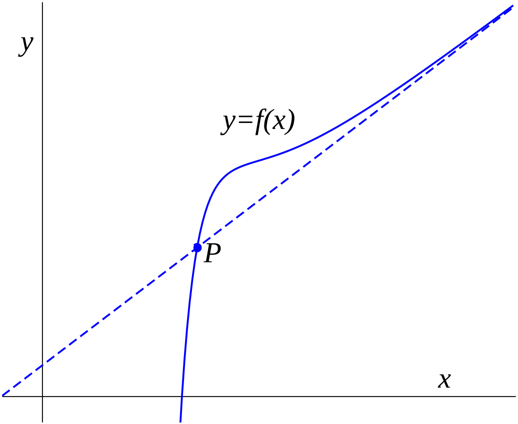
We say $f(x)$ is asymptotic to $g(x)$ as $x \to \infty$ if $\lim_{x\to\infty} [f(x) - g(x)] = 0$

There's a similar definition for $f(x)$ asymptotic to $g(x)$ as $x \to -\infty$.

Often we will be interested in if $f(x)$ is asymptotic to a line $L(x)$.

Find the coordinates of P

$$f(x) = 3x + 4 + \frac{2x - 10}{x^2}$$



Asymptotics

Find the line asymptotes of $f(x) = \sqrt{x^2 + 4x}$ as $x \to \infty$ and $x \to -\infty$.

Hint: Assume $\exists a, b \in \mathbb{R}$ s.t. $ax + b$ is asymptotic to $f(x)$ as $x \to \infty$. Write down what this means and then do the "obvious things" to find what a and b has to be.

Line asymptotes

Given function $f(x)$.

We want to see if there are $a, b \in \mathbb{R}$ such that $\lim_{x \to \infty} [f(x) - ax - b] = 0$.

Come up with a way to find a, or to tell no a works in the above limit equation. HINT: Suppose $\exists a, b \in \mathbb{R}$ s.t. the limit equation is satisfied, divide by x.

Now we have a formula for a, come up with a way to find b, or to tell no b works in the above limit equation.

Line asymptotes

Theorem

If $\lim_{x\to\infty} \frac{f(x)}{x}$ exists and equal a, and $\lim_{x\to\infty}[f(x) - ax]$ exists and equal b, then $f(x)$ is asymptotic to the line $L(x) = ax + b$ as $x \to \infty$. If any of the two limits DNE, then there are no lines asymptotic to f as $x \to \infty$

Theorem

If $\lim_{x\to-\infty} \frac{f(x)}{x}$ exists and equal a, and $\lim_{x\to-\infty}[f(x) - ax]$ exists and equal b, then $f(x)$ is asymptotic to the line $L(x) = ax + b$ as $x \to -\infty$. If any of the two limits DNE, then there are no lines asymptotic to f as $x \to -\infty$

Horizontal and Slant asymptotes

In the case line asymptotes exist as $x \to \pm\infty$, we have two cases:

When $a = 0$, we call the line asymptote a horizontal asymptote. In other words,

Horizontal asymptote

We say f has a horizontal asymptote of b_1 as $x \to \infty$ iff $\lim\limits_{x\to\infty} [f(x) - 0x] = b_1$ iff $\lim\limits_{x\to\infty} f(x) = b_1$
We say f has a horizontal asymptote of b_2 as $x \to -\infty$ iff $\lim\limits_{x\to-\infty} [f(x) - 0x] = b_2$ iff $\lim\limits_{x\to-\infty} f(x) = b_2$

When $a \neq 0$, we call the line asymptote a slant asymptote. In particular, as $x \to \infty$ or as $x \to -\infty$, slant and horizontal asymptotes are mutually exclusive.

An equation from the asymptotes

Construct a function f that satisfies all the following conditions at the same time.

- f is a rational function (this means it is a quotient of polynomials).
- The line $y = 1$ is an asymptote of the graph of f.
- The line $x = -1$ is an asymptote of the graph of f.

Periodic?

Construct a function H that has all the following properties at once:

1. The domain of H is $\mathbb{R}$
2. H is strictly increasing on $\mathbb{R}$
3. H is differentiable on $\mathbb{R}$
4. H' is periodic with a period of 2
5. H' is not constant

A weird function

The function $G(x) = xe^{1/x}$ is deceiving. To help you out:

$$G'(x) = \frac{x-1}{x} e^{1/x}, \qquad G''(x) = \frac{e^{1/x}}{x^3}$$

1. Carefully study the behaviour as $x \to \pm\infty$.
   You should find line asymptotes, but it is not easy.
2. Carefully study the behaviour as $x \to 0^+$ and $x \to 0^-$.
   The two are very different.
3. Use G' to study monotonocity.
4. Use G'' to study concavity.
5. Sketch the graph of G.