Announcements

- **Topics:** Local extrema, Rolle's Theorem, MVT, Monotonicity
- **Homework:** Watch videos 6.1 - 6.10.

Inverse differentiation review

Differentiate $\arctan(x)$. Simplify your answer so that it doesn't involve any trig or inverse trig functions.

Given interval $\mathbb{I}$ and f defined on $\mathbb{I}$.

Give the definition for "f is increasing on $\mathbb{I}$".

Theorem

Let $a < b \in \mathbb{R}$.

Let f differentiable on (a, b).

IF $\forall x \in (a, b)$, $f'(x) > 0$.

THEN f is increasing on (a, b).

What's wrong with this proof?

Theorem

Let $a < b \in \mathbb{R}$.

Let f differentiable on (a, b).

IF $\forall x \in (a, b)$, $f'(x) > 0$.

THEN f is increasing on (a, b).

Proof: Assume $\forall x \in (a, b), f'(x) > 0$.

Let $x_1, x_2 \in (a, b)$. Assume $x_2 > x_1$.

Since $f'(x_1) > 0$, we have $\lim\limits_{x_2 \to x_1} \frac{f(x_2)-f(x_1)}{x_2-x_1} > 0$.

Therefore, $\frac{f(x_2)-f(x_1)}{x_2-x_1} > 0$.

Since $x_2 - x_1 > 0$, we have $f(x_2) - f(x_1) > 0$ (i.e. $f(x_2) > f(x_1)$ as required.)

Prove positive derivative implies increasing

Prove the following theorem.

Theorem

Let $a < b \in \mathbb{R}$.

Let f differentiable on (a, b).

IF $\forall x \in (a, b)$, $f'(x) > 0$.

THEN f is increasing on (a, b).

True or false?

Theorem

Let $a < b \in \mathbb{R}$.

Let f be differentiable on (a, b).

IF $\forall x \in (a, b)$, $f'(x) > 0$.

THEN f is increasing on $[a, b]$.

Is there something you can add to the assumptions so that the stated conclusion is true?

Homework: Proof practice

Theorem

Let $a < b \in \mathbb{R}$. Let f be differentiable on (a, b).

IF $\forall x \in (a, b)$, $f'(x) \neq 0$.

THEN f is ???

Where is the local extrema?

We know the following about the function h:

- The domain of h is $(-4, 4)$.
- h is continuous on its domain.
- h is differentiable on its domain, except at 0.
- $h'(x) = 0 \quad \iff \quad x = -1$ or 1.

What can you conclude about the local extrema of h?

1. h has a local extrema at $x = -1$, and 1.
2. h has a local extrema at $x = -1, 0$, and 1.
3. h has a local extrema at $x = -4, 1, 0, 1$, and 4.
4. None of the above.

Fractional exponents

Let $g(x) = x^{2/3}(x-1)^3$.

Find local and global extrema of g on $[0, 2]$.

Let $h(x) = x^4 - 4x$.

Find local and global extrema of h on $\mathbb{R}$.

What can you conclude?

We know the following about the function f.

- f has domain $\mathbb{R}$.
- f is continuous
- $f(0) = 0$
- For every $x \in \mathbb{R}$, $f(x) \geq x$.

What can you conclude about $f'(0)$? Prove it.

Hint: Sketch the graph of f. Looking at the graph, make a conjecture.
To prove it, imitate the proof of the Local EVT from Video 5.3.

Zeroes of the derivative

If possible, construct a function f that is differentiable on $\mathbb{R}$ and such that

1. f has exactly 2 zeroes and f' has exactly 1 zero.
2. f has exactly 2 zeroes and f' has exactly 2 zeroes.
3. f has exactly 3 zeroes and f' has exactly 1 zero.
4. f has exactly 1 zero and f' has infinitely many zeroes.

How many zeroes?

Let

$$f(x) = x^2 - \cos(x)$$

How many zeroes does f have?

Let

$$g(x) = x^2 + \cos(x)$$

How many zeroes does g have?
Do this question without using any graphing utilities.

Hint: First put an upperbound on the number of zeroes both f and g has by taking second derivatives. Then analyze the sign of the first derivatives.

Roots of a polynomial

Given $n \in \mathbb{Z}^+$.

A polynomial of degree n is a function $P(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_0$ where $\forall i = 0, ..., n,\ a_i \in \mathbb{R}$ and $a_n \neq 0$.

Prove a polynomial of degree n can have at most n distinct roots.

Proving difficult identities

Prove that, for every $x \geq 0$,

$$\arcsin \frac{1-x}{1+x} + 2\arctan \sqrt{x} = \frac{\pi}{2}$$

Hint: Take derivatives.

Let $g(x) = x^3(x^2 - 4)^{1/3}$.

Find out on which intervals this function is increasing or decreasing.
Using that information, sketch its graph.

To save time, here is the first derivative:

$$g'(x) = \frac{x^2(11x^2 - 36)}{3(x^2 - 4)^{2/3}}$$

Inequalities

Prove that, for every $x \in \mathbb{R}$

$$e^x \geq 1 + x$$

Hint: When is the function $f(x) = e^x - 1 - x$ increasing or decreasing?