MAT 137Y: Calculus!
Problem Set D

1. Use Taylor series to compute the following limits

   (a) $\displaystyle\lim_{x\to 0} \frac{6\sin x - 6x + x^3}{x^5}$ (b) $\displaystyle\lim_{x\to 0} \frac{e^{x^2} - \cos(2x) - 3x^2}{x^2 \sin(x^2)}$

2. Consider the function $F(x) = \displaystyle\int_0^x e^{-t^4} dt$. It is impossible to find an "elementary antiderivative" for the function $f(t) = e^{-t^4}$, so we use series instead to understand this function.

   (a) Obtain the Taylor series of $f(t) = e^{-t^4}$ around $t = 0$.

   (b) Use the previous answer to represent the function $F(x) = \displaystyle\int_0^x e^{-t^4} dt$ as a power series.

   (c) Estimate $\displaystyle\int_0^1 e^{-t^4} dt$ with an error smaller than 0.001.

   Hint: Notice the series is alternating.

3. Consider the function $f(x) = \dfrac{1}{\sqrt{1+x}}$.

   (a) Find a formula for $f^{(n)}(x)$ and prove it.

   Suggestion: Compute a few terms, guess the pattern, then prove it by induction.

   (b) Write down an explicit formula for the Maclaurin series of $f(x)$. Let us call this series $S(x)$.

   (c) Calculate the radius of convergence of $S(x)$.

   Note: It is possible to prove that $f(x) = S(x)$ inside the interval of convergence, but it requires other versions of the Remainder Theorem. For now, just accept this without proof.

   (d) Use your answer to the previous questions to obtain the Maclaurin series for $g(x) = \arcsin x$ around $a = 0$. In which domain can you be certain that arcsin is equal to its Maclaurin series?

   Hint: What is $g'(x)$? First write the Maclaurin series for $g'(x)$ and then integrate.

(e) Give a formula for $g^{(n)}(0)$.

Hint: You do not need to take any derivatives now. This should be a quick question.

4. Use Taylor series to estimate the following quantities with an error smaller than 0.001.

   (a) $1/e$ (b) $\sin 0.3$ (c) $\ln 1.1$

5. Give an example of a power series...

   (a) whose interval of convergence is $(-42, 42)$.

   (b) whose interval of convergence is $[-5, -3]$.

   (c) whose interval of convergence is $[e, \pi)$.

   (d) whose interval of convergence is $[-1, 1]$ and which is *conditionally* convergent both at -1 and at 1.

   (e) centered at $x = -\sqrt{2}$ and whose interval of convergence is $(-\infty, \infty)$.

6. Calculate the radius of convergence, and the interval of convergence, of the following power series:

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^2} x^n$

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{3^n(n-1)}{n^3}(x-2)^n$

   (c) $\displaystyle\sum_{n=0}^{\infty} \frac{(2n)!}{n!} x^n$

   (d) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}(x+1)^n$

7. Given a power series $\sum_{n=1}^{\infty} a_n(x-1)^n$ with interval of convergence $(0, 2]$. Find the interval of convergence for the following series. Justify your answer.

   (a) $\sum_{n=1}^{\infty} 2a_n(x-2)^n$

   (b) $\sum_{n=1}^{\infty} (-2)^n(a_n)(x-2)^n$