MAT 137Y: Calculus!
Problem Set C

This problem set will test whether you understand integration techniques by asking you to integrate functions *that we have not taught you how to integrate yet.*
As for the rest of the topics in Test #3, the past problem sets and tutorials have been very comprehensive, so we are not adding any new questions.

1. We introduce two new functions

   - sinh is called "hyperbolic sine"
   - cosh is called "hyperbolic cosine"

   We know they both have domain $\mathbb{R}$ and that

   $$\sinh'(x) = \cosh(x), \qquad \cosh'(x) = \sinh(x).$$

   Note: These functions have a very precise definition and lots of interesting properties. But do not look them up yet! The point of this question is that you only need to know the properties above to compute the antiderivatives below. You do not need anything else.

   Calculate the following antiderivatives.

   (a) $\displaystyle\int \cosh(2x+1)\,dx$

   (b) $\displaystyle\int \frac{\sinh(x)}{\cosh(x)}\,dx$

   (c) $\displaystyle\int \sinh(2x)\cosh^4(2x)\,dx$

   (d) $\displaystyle\int x^2\sinh(x)\,dx$

   (e) $\displaystyle\int \sin x\ \sinh(x)\,dx$

   (f) $\displaystyle\int x\sin x\ \sinh(x)\,dx$

2. Continuing with the notation of Question 1, assume we also know that

   $$\cosh^2(x) - \sinh^2(x) = 1.$$

   Calculate the following antiderivatives. (Again, the only thing you need to know about the hyperbolic functions to solve this question is the properties listed above. Nothing more.)

   (a) $\displaystyle\int \sinh^5(x)\cosh^4(x)\,dx$

   (b) $\displaystyle\int \frac{\cosh^3(x)}{\sinh^4(x)}dx$

   (c) $\displaystyle\int \frac{\sinh^2(x)}{\cosh^4(x)}dx$

   Hint: For Question 2c, consider the function "hyperbolic tangent" $\tanh(x) = \dfrac{\sinh(x)}{\cosh(x)}$ and compute $\tanh'(x)$.

3. Let a be a postive real number.
   Suppose f is continuous and even on $[-a, a]$.
   Show $\int\limits_{-a}^{a} f(t)dt = 2\int\limits_{0}^{a} f(t)dt$ by substitition.

4. Use FTC 1 to prove that, $\forall x > 0$, $\int\limits_{0}^{x} \frac{dt}{1+t^2} + \int\limits_{0}^{\frac{1}{x}} \frac{dt}{1+t^2} = \frac{\pi}{2}$.

5. Suppse f is continuous on $\mathbb{R}$. Prove $\forall x \in \mathbb{R}$

6. Let
$$F(x) = \int_{\int\limits_{x}^{x^2} \frac{1}{1+s^4} ds}^{x^2-1} x \arctan(t) dt,$$
   find $F'(1)$. Use only FTC I and check all hypotheses.

7. Suppose f is continuous, prove $\forall x \in \mathbb{R}$, $\int\limits_{0}^{x} f(u)(x-u)du = \int\limits_{0}^{x}(\int\limits_{0}^{u} f(t)dt)du$.