MAT 137Y: Calculus!
Problem Set 8
Due on Wednesday, August 7 at 11:59 pm

Instructions:

- You will need to submit your solutions electronically. For instructions, see PS submission. Make sure you understand how to submit and that you try the system ahead of time. If you leave it for the last minute and you run into technical problems, you will be late. There are no extensions for any reason.
- **You will need to submit your answer to each question separately.**
- **Read "A note on collaboration" in here.**

IMPORTANT NOTES ON COLLABORATION

Solving a mathematical problem set has two parts:

1. The discovery phase. This is the time you spent trying to figure out how to solve the problems. You are welcome and encouraged to collaborate with other students in this phase as discussing problems with your classmates is a useful and mathematically healthy practice. We encourage group work. If you work in a group and solve a problem as a group, that's great!
2. The Write-up phase. After meeting as a group, sit on your own and take out a fresh piece of paper. Put the notes from the group work face down. Write up the solution yourself. **Be alone when you write your solutions.** If you cannot write up the solution yourself, you do not understand the solution and you'll get a nasty shock if a similar question shows up on a test.
   If you cannot write up the solution yourself, go back to the notes and work through that solution again. Put it aside and try to write it up again on your own half and hour to an hour later.
   Also, if you are not fluent in English then you're trying and solve a hard math problem and write it up in English at the same time. This is really hard!! We recommend that first you write it up in whatever language you are most comfortable in. Once you have a solution that makes sense to you and reads cleanly in that language, you're done with the math. Now translate your lovely solution into English. While we appreciate beautifully written English, we care more about your math than about your grammar and spelling.

1. For $n \geq 1$, let a_n be a real number greater than 0. For $n \geq 1$, we let $S_n = \sum_{i=1}^{n} a_i$ (so that S_n is the n-th partial sum of $\sum_{i=1}^{\infty} a_i$) and we define $T_n = \frac{1}{n}\sum_{i=1}^{n} S_i$ (the average of the first n partial sums). Suppose that the set $\{ja_j : j \geq 1\} = \{a_1, 2a_2, 3a_3, \ldots\}$ is bounded above by a real number M.

   (a) [*5 points*] Use induction to prove that for all $n \geq 1$, we have

   $$S_n - \frac{n}{n+1}T_n \leq M.$$

   (b) [*4 points*] Assume now, in addition to the above, that the sequence $\{T_n\}_{n=1}^{\infty}$ converges. Use this, together with the result from (a), to prove that the series $\sum_{n=1}^{\infty} a_n$ converges.

2. (a) [*3 points*] Show that the series

   $$\sum_{n=1}^{\infty} \left(\frac{e}{n}\right)^n$$

   converges and use an appropriate convergence test to deduce that the improper integral

   $$\int_1^{\infty} \frac{e^y}{y^y} dy$$

   converges.

   (b) [*4 points*] Using an appropriate convergence test and the fact that the improper integral in part (a) converges, show that

   $$\sum_{n=2}^{\infty} \frac{1}{(\ln(n))^{\ln(n)}}$$

   converges.

3. [*10 points*] Consider the power series

   $$\sum_{n=1}^{\infty} \frac{n+4}{7^n(n^2+11)}(x-2)^n.$$

   Determine the interval of convergence (see videos 14.1 and 14.2) of this power series. If the interval is bounded, be sure to determine whether the series converges at the endpoints.