MAT 137Y: Calculus!
Problem Set 7
Solutions

1. Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of real numbers and suppose that it converges to a real number L. We need to show that $\{a_n\}_{n=0}^{\infty}$ is Cauchy. To prove this, we fix $\epsilon > 0$. Since $a_n \to L$, we use the fixed positive real number $\epsilon/2$ in the definition of convergence. This gives us an $N \in \mathbb{N}$ with the property that for all natural numbers $n \geq N$ we have $|a_n - L| < \epsilon/2$. This is the N we will choose for our fixed ϵ to prove that the sequence is Cauchy. We need to check that

$$\forall n \in \mathbb{N}\ \forall m \in \mathbb{N}\ (n, m \geq N \implies |a_n - a_m| < \epsilon).$$

To prove this, suppose we have natural numbers $n, m \geq N$. By the way we defined N, we know that $|a_n - L| < \epsilon/2$ and that $|a_m - L| < \epsilon/2$. Therefore, we can use the triangle inequality to conclude that

$$|a_n - a_m| \leq |a_n - L| + |a_m - L| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Therefore, the sequence is Cauchy.

2. **First Solution (explicit formula for the a_n's).**

We will first prove by induction that for all $n \in \mathbb{N}$,

$$a_n = 2^{1-\frac{1}{2^{n+1}}} = 2^{\frac{2^{n+1}-1}{2^{n+1}}}.$$

For the base case, note that

$$2^{1-\frac{1}{2^{0+1}}} = 2^{1-\frac{1}{2}} = 2^{\frac{1}{2}} = \sqrt{2} = a_0,$$

as it should be. Next suppose that for a fixed $k \geq 0$, we know

$$a_k = 2^{1-\frac{1}{2^{k+1}}}.$$

Then,

$$a_{k+1} = \sqrt{2a_k} = (2a_k)^{\frac{1}{2}} = 2^{\frac{1}{2}} a_k^{\frac{1}{2}} = 2^{\frac{1}{2}} \left(2^{1-\frac{1}{2^{k+1}}}\right)^{\frac{1}{2}} = 2^{\frac{1}{2}} \cdot 2^{\frac{1}{2}-\frac{1}{2^{k+2}}} = 2^{1-\frac{1}{2^{k+2}}} = 2^{1-\frac{1}{2^{(k+1)+1}}}.$$

Therefore, the desired formula holds for a_{k+1}. Hence, by induction, we have that

$$a_n = 2^{1-\frac{1}{2^{n+1}}}$$

for all $n \in \mathbb{N}$. To see that this sequence converges to 2, simply take the limit:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} 2^{1-\frac{1}{2^{n+1}}} = 2^{\lim_{n\to\infty}\left(1-\frac{1}{2^{n+1}}\right)} = 2^1 = 2.$$

Second Solution (without explicit formula for the a_n's).

We first show that for all $x \in (0, 2)$, we have $x < \sqrt{2x} < 2$. Multiplying every term in the inequality $0 < x < 2$ by the positive number x, we obtain $0 < x^2 < 2x$. Upon taking square roots, this yields $x < \sqrt{2x}$, which is the first half of the inequality. If instead we multiply every term in the inequality $0 < x < 2$ by the positive number 2, we obtain $0 < 2x < 4$. And upon taking square roots, we obtain $\sqrt{2x} < 2$, which is the other half of the inequality. Thus, for all $x \in (0, 2)$, we have $x < \sqrt{2x} < 2$.

Now, we show that the sequence $\{a_n\}_{n=0}^{\infty}$ is increasing and bounded above by 2. To show this, we will prove that for all integers $n \geq 1$, we have $0 < a_{n-1} < a_n < 2$. We proceed by induction on n. We know that $0 < a_0 < 2$. Hence, by the above paragraph, $0 < a_0 < \sqrt{2a_0} = a_1 < 2$, which is the base case. Now, fix an integer $n \geq 1$ and suppose that $0 < a_{n-1} < a_n < 2$. Then, by the above paragraph, we know $0 < a_n < \sqrt{2a_n} = a_{n+1} < 2$. This completes the induction proof. Since the sequence in question is increasing and bounded above, the Monotone Convergence Theorem tells us that it converges to some limit L.

We now show that $L = 2$. For all integers $n \geq 1$, we know that $a_n = \sqrt{2a_{n-1}}$. Thus,

$$L = \lim_{n\to\infty} a_n = \lim_{n\to\infty} \sqrt{2a_{n-1}} = \sqrt{2L}.$$

Hence,

$$L(L-2) = L^2 - 2L = 0.$$

Thus, $L = 0$ or $L = 2$. But the limit cannot be equal to 0 since $a_0 = \sqrt{2}$ and the sequence is increasing. Therefore, $L = 2$.

3. **First Solution (Using the Limit Comparison Test)**. We consider two cases.

Case one: $b \leq 0$. We claim that in this case I converges iff $a < -1$. Note that x^a and $\frac{x^a}{1+x^b}$ are both positive and continuous functions on $[1, \infty)$. Moreover,

$$\lim_{x\to\infty} \left(\frac{x^a}{1+x^b} \Big/ x^a \right) = \lim_{x\to\infty} \frac{1}{1+x^b} = \begin{cases} \frac{1}{2} & \text{if } b = 0 \\ 1 & \text{if } b < 0 \end{cases}.$$

In either instance, the limit is finite and lies in $(0, \infty)$. Therefore, by the Limit Comparison Test, I converges iff $\int_1^\infty x^a dx$ converges. But we know that this latter integral converges iff $a < -1$.

Case two: $b > 0$. We claim that in this case I converges iff $a - b < -1$. Note that $\frac{x^a}{x^b}$ and $\frac{x^a}{1+x^b}$ are both positive and continuous functions on $[1, \infty)$. Moreover, since $b > 0$,

$$\lim_{x\to\infty} \left(\frac{x^a}{1+x^b} \Big/ \frac{x^a}{x^b} \right) = \lim_{x\to\infty} \frac{1}{x^{-b}+1} = 1 \in (0, \infty).$$

Therefore, by the Limit Comparison Test, I converges iff $\int_1^\infty x^{a-b} dx$ converges. And we know this latter integral converges iff $a - b < -1$.

Thus, putting the information from the two cases together, we conclude that I converges iff ($b \leq 0$ and $a < -1$) or ($b > 0$ and $a - b < -1$).

Second Solution (Using the Basic Comparison Test). As above, we consider two cases:

Case one: $b \leq 0$. Since $b \leq 0$, we have $1 \leq 1 + x^b \leq 2$ for all $x \geq 1$. Hence,

$$0 \leq \frac{x^a}{2} \leq \frac{x^a}{1+x^b} \leq x^a.$$

Moreover, all of the functions in the above inequality are continuous on $[1, \infty)$. By the Basic Comparison Test, if $\int_1^\infty x^a dx$ converges, then so does I. Hence, if $a < -1$, then I converges. The Basic Comparison Test also tells us that if $\int_1^\infty \frac{x^a}{2} dx$ diverges, then so does I. Thus, if $a \geq -1$, then I diverges. We conclude that I converges iff $a < -1$.

Case Two: $b > 0$. Since $b > 0$, we have $1 \leq 1 + x^b \leq 2x^b$ for all $x \geq 1$. Hence,

$$0 \leq \frac{1}{2} x^{a-b} = \frac{x^a}{2x^b} \leq \frac{x^a}{1+x^b} \leq \frac{x^a}{x^b} = x^{a-b}.$$

Moreover, all of the functions in the above inequality are continuous on $[1, \infty)$. By the Basic Comparison Test, if $\int_1^\infty x^{a-b}$ converges, then so does I. Hence, if $a - b < -1$, then I converges. The Basic Comparison

Test also tells us that if $\int_1^\infty \frac{1}{2}x^{a-b}dx$ diverges, then so does I. Thus, if $a - b \geq -1$, then I diverges. We conclude that I converges iff $a - b < -1$.

Thus, putting the information from the two cases together, we conclude that I converges iff ($b \leq 0$ and $a < -1$) or ($b > 0$ and $a - b < -1$).