MAT 137Y: Calculus!
Problem Set 7
Due on Wednesday, July 31 at 11:59 pm

Instructions:

- You will need to submit your solutions electronically. For instructions, see PS submission. Make sure you understand how to submit and that you try the system ahead of time. If you leave it for the last minute and you run into technical problems, you will be late. There are no extensions for any reason.
- **You will need to submit your answer to each question separately.**
- **Read "A note on collaboration" in here.**

IMPORTANT NOTES ON COLLABORATION

Solving a mathematical problem set has two parts:

1. The discovery phase. This is the time you spent trying to figure out how to solve the problems. You are welcome and encouraged to collaborate with other students in this phase as discussing problems with your classmates is a useful and mathematically healthy practice. We encourage group work. If you work in a group and solve a problem as a group, that's great!
2. The Write-up phase. After meeting as a group, sit on your own and take out a fresh piece of paper. Put the notes from the group work face down. Write up the solution yourself. **Be alone when you write your solutions.** If you cannot write up the solution yourself, you do not understand the solution and you'll get a nasty shock if a similar question shows up on a test.
   If you cannot write up the solution yourself, go back to the notes and work through that solution again. Put it aside and try to write it up again on your own half and hour to an hour later.
   Also, if you are not fluent in English then you're trying and solve a hard math problem and write it up in English at the same time. This is really hard!! We recommend that first you write it up in whatever language you are most comfortable in. Once you have a solution that makes sense to you and reads cleanly in that language, you're done with the math. Now translate your lovely solution into English. While we appreciate beautifully written English, we care more about your math than about your grammar and spelling.

1. In this problem, we will investigate a new property which some sequences may have. Here is a new definition.
   Definition: A sequence $\{a_n\}_{n=0}^{\infty}$ is said to be **Cauchy** iff

$$\forall \epsilon > 0 \; \exists N \in \mathbb{N} \; s.t. \; \forall n \in \mathbb{N} \; \forall m \in \mathbb{N} \; (n, m \geq N \implies |a_n - a_m| < \epsilon).$$

   Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of real numbers that converges to a number L. Show that $\{a_n\}_{n=0}^{\infty}$ must be Cauchy.

2. Let $\{a_n\}_{n=0}^{\infty}$ be a sequence defined recursively as follows:

$$\begin{cases} a_0 = \sqrt{2} \\ a_n = \sqrt{2a_{n-1}}, \text{ for } n \geq 1 \end{cases} .$$

   Show that $\{a_n\}_{n=1}^{\infty}$ converges and that its limit is 2.
   Hint: It may be helpful to first prove that for all $x \in (0, 2)$, we have $x < \sqrt{2x} < 2$.

3. Let $a, b \in \mathbb{R}$. Consider the improper integral

$$I = \int_1^{\infty} \frac{x^a}{1 + x^b} dx.$$

   For which values of a and b is I convergent? For which values of a and b is I divergent?
   Hint: You might want to consider the two cases $b > 0$ and $b \leq 0$ separately.