MAT 137Y: Calculus!
Problem Set 6 Solution.

1. In the following two parts we will define $F(x) = \int_0^x f(t)dt$. Notice since f is continuous on $\mathbb{R}$, F'(x)=f(x) by FTC 1.

   (a) Suppose $\lim_{A\to\infty} \int_0^A f(t)dt$ is finite and equals $C \in \mathbb{R}$.
   Choose $G(x) = F(x) - C - 1$. Notice G is an antiderivative of $f(x)$.
   We claim $\nexists a \in \mathbb{R}$ s.t. $G(x) = \int_a^x f(t)dt$.
   Suppose otherwise, then let a be a number s.t. $G(x) = \int_a^x f(t)dt$.
   We have $\int_0^x f(t)dt - C - 1 = \int_a^x f(t)dt$.
   $\therefore C + 1 = \int_0^x f(t)dt - \int_a^x f(t)dt = \int_0^a f(t)dt = F(a)$.
   On the otherhand, F is non-decreasing since $f(x) \geq 0$ and $\lim_{x\to\infty} F(x) = C$ by assumption. This is a contradiction and therefore $\nexists a \in \mathbb{R}$ s.t. $G(x) = \int_a^x f(t)dt$.
   In the case that $\lim_{A\to-\infty} \int_A^0 f(t)dt$ is finite and equals C we can choose $G(x) = F(x) + C + 1$ and argue similarly.

   (b) Note the assumptions are equivalent to saying $\lim_{x\to\infty} F(x) = \infty$ and $\lim_{x\to-\infty} F(x) = -\infty$.
   Given $G(x)$ antiderivative of $f(x)$. We divide into three cases.
   Case 1: $G(0) = 0$
   If G(0) = 0 then $G(x) = \int_0^x f(t)dt$ since both are equal to 0 at $x = 0$ and have the same derivative.
   Case 2: $G(0) > 0$
   Since $\lim_{x\to-\infty} F(x) = -\infty$, by definition $\exists M \in \mathbb{R}$ s.t. $F(M) < -G(0)$. Note M is necessarily negative since F is non-decreasing and $F(0) = 0$.
   Since $F(x)$ is differentiable it is also continuous on $\mathbb{R}$.
   Moreover $F(M) < -G(0) < F(0) = 0$. So by IVT on $[M, 0]$, we can choose $a \in [M, 0]$ s.t. $F(a) = -G(0)$.
   We claim $G(x) = \int_a^x f(t)dt$.
   This is because $\int_a^0 f(t)dt = -F(a) = G(0)$. So the two functions agree in value at $x = 0$ and have the same derivative, and are therefore the same functions.
   Case 3: $G(0) < 0$
   A similar argument to case 2 would work.

2. Define $G(x) = \int_0^x f(t)dt$. Since $f(x)$ is continuous on $\mathbb{R}$, by FTC 1 $G'(x) = f(x)$. Notice the assumptions in the problem simply says $G(1) = 1$ and $G'(1) = 2$.
   We prove by induction that $\forall n \in \mathbb{N}$, $F_n(1) = 1$ and $F_n'(1) = 2^{n+1} - 1$. We only need to prove the second part, but the first part will be used in the induction proof of the second part.

Base Case: $F_1(1) = 1$ and $F_1'(1) = 3$.

$F_1(1) = \int_0^1 (1)f(t)dt = 1$ by assumption.

$F_1(x) = x\int_0^x f(t)dt = xG(x)$.

$$\begin{aligned} \therefore F_1'(1) &= G(1) + (1)G'(1) \\ &= 1 + 2 \\ &= 3. \end{aligned}$$

Inductive step: We show $\forall n \in \mathbb{N}$, "$F_n(1) = 1$ and $F_n'(1) = 2^{n+1} - 1$" $\implies$ "$F_{n+1}(1) = 1$ and $F_{n+1}'(1) = 2^{n+2} - 1$."

We have

$$\begin{aligned} F_{n+1}(x) &= \int_0^{F_n(x)} xf(t)dt \\ &= x\int_0^{F_n(x)} f(t)dt \\ &= xG(F_n(x)). \end{aligned}$$

$\therefore F_{n+1}(1) = (1)G(F_n(1)) = G(1) = 1$.

Differentiating, using chain rule and product rule, we have

$$\begin{aligned} F_{n+1}'(1) &= G(F_n(1)) + (1)G'(F_n(1))F_n'(1) \\ &= 1 + G'(1)(2) \\ &= 1 + (2^{n+1} - 1)(2) \\ &= 2^{n+2} - 1. \end{aligned}$$

By induction, we conclude that $\forall n \in \mathbb{N}$, $F_n'(1) = 2^{n+1} - 1$.