MAT 137Y: Calculus!
Problem Set 6.
Due on Wednesday, July 17 at midnight

Instructions:

- You need to submit this assignment electronically. If you write your solutions by hand, you may scan them using your phone (some free scanning apps are Scanbot and CamScanner). There are also free scanners in the Robarts library for you to use.
- For instructions on how to submit online, see here.
- **You will need to submit the answer to each question separately.**
- **Read "A note on collaboration" in .**

IMPORTANT NOTES ON COLLABORATION

Solving a mathematical problem set has two parts:

1. The discovery phase. This is the time you spent trying to figure out how to solve the problems. You are welcome and encouraged to collaborate with other students in this phase as discussing problems with your classmates is a useful and mathematically healthy practice. We encourage group work. If you work in a group and solve a problem as a group, that's great!
2. The Write-up phase. After meeting as a group, sit on your own and take out a fresh piece of paper. Put the notes from the group work face down. Write up the solution yourself. **Be alone when you write your solutions.** If you cannot write up the solution yourself, you do not understand the solution and you'll get a nasty shock if a similar question shows up on a test.
   If you cannot write up the solution yourself, go back to the notes and work through that solution again. Put it aside and try to write it up again on your own half and hour to an hour later.
   Also, if you are not fluent in English then you're trying and solve a hard math problem and write it up in English at the same time. This is really hard!! We recommend that first you write it up in whatever language you are most comfortable in. Once you have a solution that makes sense to you and reads cleanly in that language, you're done with the math. Now translate your lovely solution into English. While we appreciate beautifully written English, we care more about your math than about your grammar and spelling.

1. Let f be continuous on $\mathbb{R}$.

   Due to FTC I, we know that a function of the form* $F(x) = \int_a^x f(t)dt$ is always an antiderivative of $f(x)$. In this question you will investigate whether all antiderivatives of $f(x)$ can be expressed in this form*. For simplicity, let us further assume f is **non-negative** (i.e. $\forall x \in \mathbb{R}, f(x) \geq 0$).

   (a) Suppose $\lim_{A\to\infty} \int_0^A f(t)dt$ **or** $\lim_{A\to-\infty} \int_A^0 f(t)dt$ is finite, show there is an antiderivative $G(x)$ of $f(x)$ which does not equal $\int_a^x f(t)dt$ for any $a \in \mathbb{R}$.

   (b) Supppse $\lim_{A\to\infty} \int_0^A f(t)dt = \infty$ **and** $\lim_{A\to-\infty} \int_A^0 f(t)dt = \infty$, show for any antiderivative $G(x)$ of $f(x)$, $\exists a \in \mathbb{R}$ s.t. $G(x) = \int_a^x f(t)dt$.

   Hint: Think about whether antiderivatives of $f(x)$ need to have zeroes.

2. Let f be continuous on $\mathbb{R}$.

   Consider the following sequence of functions:

   $F_1(x) = \int_0^x xf(t)dt$

   $F_2(x) = \int_0^{(\int_0^x xf(t)dt)} xf(t)dt$

   $F_3(x) = \int_0^{(\int_0^{(\int_0^x xf(t)dt)} xf(t)dt)} xf(t)dt$ ...

   Given $\int_0^1 f(t)dt = 1$ and $f(1) = 2$.

   Evaluate $F_n^{'}(1)$ for each $n \in \mathbb{N}$.

   Make sure to justify your answer and only use FTC I in its most basic form to evaluate the derivative of integrals.