MAT 137Y: Calculus!
Problem Set 5 Solutions

1. (a) The length of each sub-interval is $\Delta x = \frac{b-a}{n}$

   (b) $x_i = a + i\frac{b-a}{n}$

   (c) $S^{\star}_{P_n}(f) = \sum_{i=1}^{n}((a+i\frac{b-a}{n})^3 - (a+i\frac{b-a}{n}))\frac{b-a}{n}$

   (d)

$$
\begin{aligned}
S^{\star}_{P_n}(f) &= \sum_{i=1}^{n} \frac{b-a}{n}[(a+i\frac{b-a}{n})^3 - (a+i\frac{b-a}{n})] \\
&= \sum_{i=1}^{n} \frac{b-a}{n}[a^3 + 3a^2(\frac{b-a}{n})i + 3a(\frac{b-a}{n})^2 i^2 + (\frac{b-a}{n})^3 i^3 - a - i(\frac{b-a}{n})] \\
&= \frac{b-a}{n}[(a^3-a)\sum_{i=1}^{n} 1 + (3a^2(\frac{b-a}{n}) - (\frac{b-a}{n}))\sum_{i=1}^{n} i + 3a(\frac{b-a}{n})^2 \sum_{i=1}^{n} i^2 + (\frac{b-a}{n})^3 \sum_{i=1}^{n} i^3] \\
&= \frac{b-a}{n}[(a^3-a)(n) + (3a^2-1)(\frac{b-a}{n})\frac{(n)(n+1)}{2} + 3a(\frac{b-a}{n})^2 \frac{(n)(n+1)(2n+1)}{6} + (\frac{b-a}{n})^3 \frac{(n)^2(n+1)}{4} \\
&= (b-a)(a^3-a) + (3a^2-1)(b-a)^2\frac{n+1}{2n} + 3a(b-a)^3\frac{(n+1)(2n+1)}{6n^2} + (b-a)^4\frac{(n+1)^2}{4n^2}
\end{aligned}
$$

   (e) Taking appropriate limits via either L'Hopital's Rule or factoring out terms of highest power, we obtain

   $\lim_{n\to\infty} S^{\star}_{P_n}(f) = (b-a)(a^3-a) + \frac{(3a^2-1)(b-a)^2}{2} + a(b-a)^3 + \frac{(b-a)^4}{4}$

2. (a) Let $P = \{x_0 = 0, x_1, \ldots x_n = 1\}$ be a partition of $[0,1]$.
   Let i be an integer between 1 and n.
   By definition $m_i = \inf_{x\in[x_{i-1},x_i]} g(x)$.
   Notice $\forall x \in [x_{i-1}, x_i] \subseteq [0,1], x^3 - x \geq -1$. Therefore, $\forall x \in [x_{i-1}, x_i], g(x) \geq -1$ and so -1 is a lower bound of $\{g(x) | x \in [x_{i-1}, x_i]\}$.
   Moreover, since $\mathbb{Q}$ is dense in $[0,1]$. There is a rational number in $[x_{i-1}, x_i]$ and so $-1 \in \{g(x) | x \in [x_{i-1}, x_i]\}$. Therefore, -1 is the least upperbound of $\{g(x) | x \in [x_{i-1}, x_i]\}$. In otherwords, $m_i = -1$.
   Therefore,

$$
\begin{aligned}
L(g,P) &= \sum_{i=1}^{n} m_i \Delta x_i \\
&= \sum_{i=1}^{n}(-1)(x_i - x_{i-1}) \\
&= -\sum_{i=1}^{n}(x_i - x_{i-1}) \\
&= -1 \text{ by telescoping.}
\end{aligned}
$$

   By definition, $\underline{I_0^1}(g) = \inf\{L(g,P) | P \text{ is a partition of } [0,1]\}$.
   Since for any partition P of $[0,1]$, $L(g,P) = -1$ from the above computation, we conclude $\underline{I_0^1}(g) = -1$.

   (b) Let $f(x) = x^3 - x$ on [0,1].
   Let $P = \{x_0 = 0, x_1, \ldots x_n = 1\}$ be a partition of $[0,1]$.
   Let i be an integer between 1 and n.
   By definition $M_i = \sup_{x\in[x_{i-1},x_i]} g(x)$.
   Let $M_i' = \sup_{x\in[x_{i-1},x_i]} f(x)$.

We claim $M_i' = M_i$.

Since f is continuous, there exists $x' \in [x_{i-1}, x_i]$ s.t. $f(x') = M_i'$. Since the irrationals are dense in $\mathbb{R}$, there are irrational numbers arbitrarily close to x'. By continuity of f again, it should be clear that $M_i' = M_i$.

Therefore, $U(f,P) = U(g,P)$ and so $\overline{I_0^1}(f) = \overline{I_0^1}(g)$.

Since f is integrable, we know $\overline{I_0^1}(f) = \int_0^1 f(x)dx = -\frac{1}{4}$ from the formula in Q1.

Therefore, $\overline{I_0^1}(g) = -\frac{1}{4}$.

(c) g is not integrable on $[0,1]$ since $\overline{I_0^1}(g) \neq \underline{I_0^1}(g)$.

3. **Proof:** FIrst let us bound $U_P(f) - L_P(f)$ for a specific class of partitions of $[a,b]$ (see hint). Let $n \in \mathbb{N}$.

Let $P_n = \left\{a, a + \frac{b-a}{n}, a + 2\frac{b-a}{n}, ..., b\right\}$.

Let i be an integer between 1 and n.

Let $m_i = \inf\limits_{x \in [x_{i-1}, x_i]} f(x)$ and $M_i = \sup\limits_{x \in [x_{i-1}, x_i]} f(x)$.

Since f is C-pink, it is continuous (see proof on last page, we did not take off marks if you did not have this proof). Therefore by EVT, $\exists u_i, v_i \in [x_{i-1}, x_i]$ s.t. $f(u_i) = m_i$ and $f(v_i) = M_i$.

Using the fact that f is C-pink, we have $M_i - m_i = |f(v_i) - f(u_i)| \leq C|v_i - u_i| \leq C|x_i - x_{i-1}| = C\frac{b-a}{n}$.

Therefore,

$$\begin{aligned} U_{P_n}(f) - L_{P_n}(f) &= \sum_{i=1}^{n} M_i \frac{b-a}{n} - \sum_{i=1}^{n} m_i \frac{b-a}{n} \\ &= \sum_{i=1}^{n} (M_i - m_i) \frac{b-a}{n} \\ &= \sum_{i=1}^{n} C(\frac{b-a}{n})^2 \\ &= C\frac{(b-a)^2}{n}. \end{aligned}$$

We will now prove f is integrable on $[a,b]$ using the ϵ-reformulation definition.

Given $\epsilon > 0$.

Choose $N \in \mathbb{N}$ s.t. $N > \frac{C(b-a)^2}{\epsilon}$. Consider P_N as above.

Then

$$\begin{aligned} U_{P_n}(f) - L_{P_n}(f) &\leq C\frac{(b-a)^2}{N} \\ &\leq C\frac{(b-a)^2}{\frac{c(b-a)^2}{\epsilon}} \\ &= \epsilon. \end{aligned}$$

Since we can do this for any $\epsilon > 0$. We conclude that f is integrable on $[a,b]$.

4. (a) **Proof:** Let $M_1 = \sup\limits_{x \in [a,b]} f(x)$ and $M_2 = \sup\limits_{x \in [a,b]} g(x)$.

Let $x \in [a,b]$.

We know $M_1 \geq f(x)$ and $M_2 \geq g(x)$.

Therefore, $M_1 + M_2 \geq f(x) + g(x)$.

Since this is true $\forall x \in [a,b]$, $M_1 + M_2$ is an upperbound to $\{f(x) + g(x) | x \in [a,b]\}$.

Since sup is the **least** upperbound, we conclude $M_1 + M_2 \geq \sup\limits_{x \in [a,b]} f(x) + g(x)$.

(b) **Proof:** For any partition $P = \{x_0, x_1, ..., x_n\}$ of $[a,b]$. We have:

$$
\begin{aligned}
U_P(f+g) &= \sum_{i=1}^{n} M_{f+g,i}\Delta x_i \\
&\leq \sum_{i=1}^{n} (M_{f,i} + M_{g,i})\Delta x_i \text{ by Q4(a) and } \Delta x_i > 0 \\
&= U_P(f) + U_P(g),
\end{aligned}
$$

where $M_{f+g,i}, M_{f,i}$, and $M_{g,i}$ are M_i as defined above for the function in the index.

By definition, for any $\epsilon > 0$, there are partitions P and Q s.t. $U_P(f) - \overline{I}_a^b(f) < \frac{\epsilon}{2}$ and $U_Q(g) - \overline{I}_a^B(g) < \frac{\epsilon}{2}$. So we have:

$$
\begin{aligned}
\overline{I}_a^b(f+g) &\leq U_{P\cup Q}(f+g) \\
&\leq U_{P\cup Q}(f) + U_{P\cup Q}(g) \\
&\leq U_P(f) + U_Q(g) \\
&\leq \overline{I}_a^b(f) - \frac{\epsilon}{2} + \overline{I}_a^b(g) - \frac{\epsilon}{2} \\
&= \overline{I}_a^b + \overline{I}_a^b(g) - \epsilon.
\end{aligned}
$$

Since this is true for every $\epsilon > 0$, we conclude $\overline{I}_a^b(f+g) \leq \overline{I}_a^b + \overline{I}_a^b(g)$. Similarly, we have $m_{f+g,i} \geq m_{f,i} + m_{g,i}$. Using the same arguments as above with lower integrals we can conclude $\underline{I}_a^b(f) + \underline{I}_a^b(g) \leq \underline{I}_a^b(f+g)$.

Putting all this together, we have

$$
\underline{I}_a^b(f) + \underline{I}_a^b(g) \leq \underline{I}_a^b(f+g) \leq \overline{I}_a^b(f+g) \leq \overline{I}_a^b(f) + \overline{I}_a^b(g).^*
$$

To conclude, since f and g are integrable on $[a, b]$, we have $\underline{I}_a^b(f) + \underline{I}_a^b(g) = \overline{I}_a^b(f) + \overline{I}_a^b(g)$. Therefore, all inequalities in * are equalities, and we have $\underline{I}_a^b(f+g) = \overline{I}_a^b(f+g)$.

Let $C > 0$.

Claim: f is C-pink on $[a,b] \implies f$ is continuous on$[a,b]$.

Proof: Assume f is C-pink.

Let $d \in [a,b]$.

Given $\epsilon > 0$.

Choose $\delta = \frac{\epsilon}{C}$.

Assume $x \in [a,b]$ and $|x-d| < \delta$.

Then $|f(x) - f(d)| \leq C|x-d| < C\delta = \epsilon$ since f is C-pink.