MAT 137Y: Calculus!
Problem Set 5.
Due on Wednesday, July 10 at midnight

Instructions:

- You need to submit this assignment electronically. If you write your solutions by hand, you may scan them using your phone (some free scanning apps are Scanbot and CamScanner). There are also free scanners in the Robarts library for you to use.
- For instructions on how to submit online, see here.
- **You will need to submit the answer to each question separately.**
- **Read "A note on collaboration" in .**

IMPORTANT NOTES ON COLLABORATION

Solving a mathematical problem set has two parts:

1. The discovery phase. This is the time you spent trying to figure out how to solve the problems. You are welcome and encouraged to collaborate with other students in this phase as discussing problems with your classmates is a useful and mathematically healthy practice. We encourage group work. If you work in a group and solve a problem as a group, that's great!
2. The Write-up phase. After meeting as a group, sit on your own and take out a fresh piece of paper. Put the notes from the group work face down. Write up the solution yourself. **Be alone when you write your solutions.** If you cannot write up the solution yourself, you do not understand the solution and you'll get a nasty shock if a similar question shows up on a test.
   If you cannot write up the solution yourself, go back to the notes and work through that solution again. Put it aside and try to write it up again on your own half and hour to an hour later.
   Also, if you are not fluent in English then you're trying and solve a hard math problem and write it up in English at the same time. This is really hard!! We recommend that first you write it up in whatever language you are most comfortable in. Once you have a solution that makes sense to you and reads cleanly in that language, you're done with the math. Now translate your lovely solution into English. While we appreciate beautifully written English, we care more about your math than about your grammar and spelling.

1. Let $a, b \in \mathbb{R}$. We will compute the integral $I = \int\limits_a^b (x^3 - x)dx$ in this problem. Since f is continuous on $[a, b]$, we know it's integrable and hence the value of the integral can be computed as a limit of Riemann sums (see video 7.11).

   Let $n \in \mathbb{N}$. Let P_n be the partition dividing $[a, b]$ into n equal sub-intervals. Notice that $\lim\limits_{n\to\infty} ||P_n|| = 0$. Hence, we can write $I = \lim\limits_{n\to\infty} S^{\star}_{P_n}(f)$ where $S^{\star}_{P_n}(f)$ is any Riemann sum for f and P_n. In particular, to make things simpler, we will use Riemann sums always choosing the right end-point to evaluate f on each subinterval.

   (a) What is the length of each sub-interval in P_n?

   (b) Let us write $P_n = \{x_0, x_1, ..., x_n\}$. Find a formula for x_i in terms of i and n.

   (c) Since we are using the right-endpoint, it means we are picking $x_i^{\star} = x_i$. Use your answer from (a) and (b) to obtain an expression for $S^{\star}_{P_n}(f)$ in the form of a sum with sigma notation.

   (d) Using the fomulas $\sum\limits_{i=1}^{N} i = \frac{N(N+1)}{2}$, $\sum\limits_{i=1}^{N} i^2 = \frac{N(N+1)(2N+1)}{6}$, $\sum\limits_{i=1}^{N} i^3 = \frac{N^2(N+1)^2}{4}$ when needed, add up up the expression you got in (c) to obtain a nice, compact formula for $S^{\star}_{P_n}(f)$ without any sums or sigma symbols.

   (e) Calculate $\lim\limits_{n\to\infty} S^{\star}_{P_n}(f)$. This number will be the exact value of $\int\limits_a^b (x^3 - x)dx$.

2. Define a function $g(x) : [0, 1] \to \mathbb{R}$ by the following formula:

$$g(x) = \begin{cases} -1 & if \quad x \in \mathbb{Q} \\ x^3 - x & if \quad x \notin \mathbb{Q}. \end{cases}$$

   (a) What is $\underline{I_a^b}(g)$? Justify your answer from the definition.

   (b) What is $\overline{I_a^b}(g)$? Justify your answer from the definition. Hint: Use (1).

   (c) Is g integrable on $[0, 1]$? Justify your answer.

3. Let $a, b \in \mathbb{R}$. We learned that if f is continuous on $[a, b]$ then f is integrable on $[a, b]$. We did not look at the proof of this, which is quite involved, in class. You will prove a weaker version of this theorem in this question. First we need a definition:

   Definition: Let $c > 0$. Given $f : [a, b] \to \mathbb{R}$. We say f is **c-pink** iff "$\forall x \in [a, b], \forall y \in [a, b], |f(x) - f(y)| \leq c|x - y|$".

   Let $c > 0$. Fix a c-pink function $f : [a, b] \to \mathbb{R}$.

   Prove that f is integrable on $[a, b]$.

   Hint: Let $n \in \mathbb{N}$. Let P_n be the partition dividing $[a, b]$ into n equal sub-intervals. Using the fact that f is c-pink, what simple expression can you conclude $U(f, P_n) - L(f, P_n)$ is less than? Conclude with the $\epsilon-$reformulation of integrability[1].

4. Let $a, b \in \mathbb{R}$.

   (a) Given $f, g : [a, b] \to \mathbb{R}$. Prove $\sup\limits_{x\in[a,b]} f(x) + \sup\limits_{x\in[a,b]} g(x) \geq \sup\limits_{x\in[a,b]} (f(x) + g(x))$ from the definition of sup.

   (b) Given $f, g : [a, b] \to \mathbb{R}$. Prove from definition that if f and g are integrable on $[a, b]$ then $f + g$ is integrable on $[a, b]$. Hint: You will need to use (a) and a similar result. You do not need to prove the other result because the proof will be very similar to what you wrote for (a).

[1] The ϵ-reformualtion of integrability is an alternative definition of integrability which is equivalent to the main definition we use. It says f is integrable on $[a, b]$ iff $\forall \epsilon > 0$, $\exists$ partition P of $[a, b]$ s.t. $U(f, P) - L(f, P) < \epsilon$. You may use this definition in place of the main definition whenever needed.