MAT 137Y: Calculus!
Practice problems for Playlist 7

Our textbook introduces the definition of integral in a very advanced way. You can read about it in Sections 5.1–5.4, but it is beyond the scope of this course, and perhaps more appropriate for MAT157. The practice problems from those sections, while interesting, may be a bit too much.

Instead of using those practice problems, we have created this problem list as practice problems for Playlist 7

1. Compute these sums

   (a) $\displaystyle\sum_{i=2}^{4} i^j$ (b) $\displaystyle\sum_{j=2}^{4} i^j$ (c) $\displaystyle\sum_{k=2}^{4} i^j$ (d) $\displaystyle\sum_{n=0}^{100} \sin\left(\frac{\pi n}{2}\right)$

2. For each positive integer N, let us define $H_N = \displaystyle\sum_{i=1}^{N} \frac{1}{i}$. This is a new sequence of terms (called the *harmonic sums*), that appears often in mathematics, just like $N!$, for example. Write each of the following sums in terms of harmonic sums:

   (a) $\displaystyle\sum_{i=10}^{20} \frac{1}{i}$

   (b) $\displaystyle\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \ldots + \frac{1}{100}$

   (c) $\displaystyle 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \ldots + \frac{1}{99}$

   (d) $\displaystyle\sum_{i=1}^{100} \frac{(-1)^{i+1}}{i}$

3. Calculate the supremum, infimum, maximum, and minimum (if they exist) of the following sets:

   (a) $\left\{x \in \mathbb{R} \mid x^2 \leq 2\right\}$ (b) $\left\{x \in \mathbb{Q} \mid x^2 \leq 2\right\}$ (c) $\left\{x \in \mathbb{Z} \mid x^2 \leq 2\right\}$

4. Consider the function f defined by $f(x) = \tan x$. If possible, construct an interval I contained in the domain of f such that...

   (a) ... f has a maximum and a minimum on I.
   (b) ... f has a supremum but not a maximum on I.
   (c) ... f has no supremum and no infimum on I.
   (d) ... f has an infimum but not a supremum on I.
   (e) ... f has a minimum, but not an infimum on I.

5. Consider the function f defined by $f(x) = x$.
   (a) Find a partition P of $[0, 1]$ such that $L_P(f) = 0.24$.
   (b) Let $c \in [0, 0.5)$. Find a partition P of $[0, 1]$ such that $L_P(f) = c$.
   (c) Prove that for every partition P of $[0, 1]$, $L_P(f) + U_P(f) = 1$.
6. Let $a < b$. Let f be a bounded function on $[a, b]$. Given a partition P of $[a, b]$, let us call $\ell_P(f)$ the Riemann sum for f and P when we choose the left end-point for each subinterval.
   (a) Find a simple property f must satisfy to guarantee that $L_P(f) = \ell_P(f)$.
   (b) Find a simple property f must satisfy to guarantee that $U_P(f) = \ell_P(f)$.
7. Is it possible to have a function f bounded on $[0, 1]$ and two partitions P and Q of $[0, 1]$ such that $P \subseteq Q$, $L_P(f) = L_Q(f)$, and $U_Q(f) \neq U_P(f)$? Either give an example, or prove it is impossible.
8. Let $a < b$. Let f be a bounded function on $[a.b]$. Prove that the following two statements are *equivalent*:
   (a) f is integrable on $[a, b]$.
   (b) There exists a UNIQUE number I such that for every partition P of $[a, b]$, $L_P(f) \leq I \leq U_P(f)$.