Today's topics and news

- Topic: Improper integrals
- **Homework for Friday:** Watch videos 13.1 - 13.9.

Recall the definitions

1. Let f be a bounded, continuous function on $[c, \infty)$. How do we define the improper integral

$$\int_c^\infty f(x)dx\,?$$

2. Let f be a continuous function on $(a, b]$. How do we define the improper integral

$$\int_a^b f(x)dx\,?$$

Computation

Calculate, using the definition of improper integral

$$\int_1^\infty \frac{1}{x^2 + x} dx$$

The most important improper integrals

Use the definition of improper integral to determine for which values of $p \in \mathbb{R}$ each of the following improper integrals converges.

1. $\displaystyle\int_0^1 \frac{1}{x^p}\,dx$
2. $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx$
3. $\displaystyle\int_0^\infty \frac{1}{x^p}\,dx$

Warm-up

Is there a difference between:

$$\lim_{x\to\infty} x - \lim_{x\to\infty} x$$

and

$$\lim_{x\to\infty} (x - x)?$$

A "simple" integral

What is $\displaystyle\int_{-1}^{1} \frac{1}{x}\, dx$?

1. $\displaystyle\int_{-1}^{1} \frac{1}{x}\, dx = (\ln|x|)\Big|_{-1}^{1} = \ln|1| - \ln|-1| = 0$
2. $\displaystyle\int_{-1}^{1} \frac{1}{x}\, dx = 0$ because $f(x) = \dfrac{1}{x}$ is an odd function.
3. $\displaystyle\int_{-1}^{1} \frac{1}{x}\, dx$ is divergent.

Computation

Does $\int_0^\infty \frac{1}{x^2-3x+2}$ converge or diverge? Check with a computation.

A simple BCT application

We want to determine whether $\displaystyle\int_1^\infty \frac{1}{x+e^x}\,dx$ is convergent or divergent.

We can try at least two comparisons:

1. Compare $\dfrac{1}{x}$ and $\dfrac{1}{x+e^x}$.
2. Compare $\dfrac{1}{e^x}$ and $\dfrac{1}{x+e^x}$.

Try both. What can you conclude from each one of them?

True or False

Let $a \in \mathbb{R}$.

Let f and g be continuous functions on $[a, \infty)$.

Assume that $\boxed{\forall x \geq a, \quad 0 \leq f(x) \leq g(x)}$.

What can we conclude?

1. IF $\int_a^\infty f(x)dx$ is convergent, THEN $\int_a^\infty g(x)dx$ is convergent.
2. IF $\int_a^\infty f(x)dx = \infty$, THEN $\int_a^\infty g(x)dx = \infty$.
3. IF $\int_a^\infty g(x)dx$ is convergent, THEN $\int_a^\infty f(x)dx$ is convergent.
4. IF $\int_a^\infty g(x)dx = \infty$, THEN $\int_a^\infty f(x)dx = \infty$.

True or False - Part II

Let $a \in \mathbb{R}$.

Let f and g be continuous functions on $[a, \infty)$.

Assume that $\boxed{\forall x \geq a, \quad f(x) \leq g(x)}$.

What can we conclude?

1. IF $\int_a^\infty f(x)dx$ is convergent, THEN $\int_a^\infty g(x)dx$ is convergent.
2. IF $\int_a^\infty f(x)dx = \infty$, THEN $\int_a^\infty g(x)dx = \infty$.
3. IF $\int_a^\infty g(x)dx$ is convergent, THEN $\int_a^\infty f(x)dx$ is convergent.
4. IF $\int_a^\infty g(x)dx = \infty$, THEN $\int_a^\infty f(x)dx = \infty$.

True or False - Part III

Let $a \in \mathbb{R}$.

Let f and g be continuous functions on $[a, \infty)$.

Assume that $\boxed{\exists M \geq a \text{ s.t. } \forall x \geq M, \quad 0 \geq f(x) \geq g(x)}$.

What can we conclude?

1. IF $\int_a^{\infty} f(x)dx$ is convergent, THEN $\int_a^{\infty} g(x)dx$ is convergent.
2. IF $\int_a^{\infty} f(x)dx = -\infty$, THEN $\int_a^{\infty} g(x)dx = -\infty$.
3. IF $\int_a^{\infty} g(x)dx$ is convergent, THEN $\int_a^{\infty} f(x)dx$ is convergent.
4. IF $\int_a^{\infty} g(x)dx = -\infty$, THEN $\int_a^{\infty} f(x)dx = -\infty$.

BCT calculations

Use the BCT to determine whether each of the following is convergent or divergent

1. $\displaystyle\int_1^\infty \frac{1+\cos^2 x}{x^{2/3}}\,dx$
2. $\displaystyle\int_1^\infty \frac{1+\cos^2 x}{x^{4/3}}\,dx$
3. $\displaystyle\int_0^\infty \frac{\arctan x^2}{1+e^x}\,dx$
4. $\displaystyle\int_0^\infty e^{-x^2}\,dx$
5. $\displaystyle\int_2^\infty \frac{(\ln x)^{10}}{x^2}\,dx$
6. $\displaystyle\int_2^\infty \frac{x^2+1}{x^4-2}\,dx$

What can you conclude?

Let $a \in \mathbb{R}$. Let f be a continuous, positive function on $[a, \infty)$.

In each of the following cases, what can you conclude about $\int_a^\infty f(x)dx$?

Is it convergent, divergent, or we do not know?

1. $\forall b \geq a, \ \exists M \in \mathbb{R}$ s.t. $\displaystyle\int_a^b f(x)dx \leq M.$
2. $\exists M \in \mathbb{R}$ s.t. $\forall b \geq a, \ \displaystyle\int_a^b f(x)dx \leq M.$
3. $\exists M > 0$ s.t. $\forall x \geq a, \ f(x) \leq M.$
4. $\exists M > 0$ s.t. $\forall x \geq a, \ f(x) \geq M.$

A generalization of LCT

This is the theorem you have learned:

Theorem (Limit-Comparison Test)

Let $a \in \mathbb{R}$. Let f and g be positive, continuous functions on $[a, \infty)$.

- IF the limit $L = \lim_{x \to \infty} \dfrac{f(x)}{g(x)}$ exists and $L > 0$.
- THEN $\displaystyle\int_a^\infty f(x)dx$ and $\displaystyle\int_a^\infty g(x)dx$ are both convergent or both divergent.

What if we change the hypothesis to $L = 0$?

1. Write down the new version of this theorem (different conclusion).
2. Prove it.

Hint: If $\lim_{x \to \infty} \dfrac{f(x)}{g(x)} = 0$, what is larger? $f(x)$ or $g(x)$?

Comparison tests for type 2 improper integrals

In the videos, you learned BCT and LCT for type-1 improper integrals.

1. Write the statement of the (standard and eventual) BCT for type-2 improper integrals.
2. Write the statement of the LCT for type-2 improper integrals.
3. Construct an example that you can prove is convergent thanks to one of these theorems.
4. Construct an example that you can prove is divergent thanks to one of these theorems.

Convergent or divergent?

1. $$\int_1^\infty \frac{x^3+2x+7}{x^5+11x^4+1}\,dx$$
2. $$\int_1^\infty \frac{1}{\sqrt{x^2+x+1}}\,dx$$
3. $$\int_0^1 \frac{3\cos x}{x+\sqrt{x}}\,dx$$
4. $$\int_0^1 \cot x\,dx$$
5. $$\int_0^1 \frac{\sin x}{x^{3/2}}\,dx$$
6. $$\int_0^\infty e^{-x^2}\,dx$$
7. $$\int_2^\infty \frac{(\ln x)^{10}}{x^2}\,dx$$
8. $$\int_0^1 \frac{\arcsin x}{x^{3/2}}\,dx$$