Today's topics and news

- Topic: Sequences, MCT, Big Theorem
- **Homework for Wednesday:** Watch videos 12.1 - 12.10.
- **Homework for Friday:** Watch videos 13.1 - 13.9.

Definition of limit of a sequence

Let $\{a_n\}_{n=0}^{\infty}$ be a sequence. Let $L \in \mathbb{R}$.
Which statements are equivalent to "$\{a_n\}_{n=0}^{\infty} \longrightarrow L$"?

1. $\forall \varepsilon > 0,\ \exists n_0 \in \mathbb{N}$ s.t. $\forall n \in \mathbb{N},\quad n \geq n_0 \implies |L - a_n| < \varepsilon.$
2. $\forall \varepsilon > 0,\ \exists n_0 \in \mathbb{N}$ s.t. $\forall n \in \mathbb{N},\quad n > n_0 \implies |L - a_n| < \varepsilon.$
3. $\forall \varepsilon > 0,\ \exists n_0 \in \mathbb{R}$ s.t. $\forall n \in \mathbb{N},\quad n \geq n_0 \implies |L - a_n| < \varepsilon.$
4. $\forall \varepsilon > 0,\ \exists n_0 \in \mathbb{N}$ s.t. $\forall n \in \mathbb{R},\quad n \geq n_0 \implies |L - a_n| < \varepsilon.$
5. $\forall \varepsilon > 0,\ \exists n_0 \in \mathbb{N}$ s.t. $\forall n \in \mathbb{N},\quad n \geq n_0 \implies |L - a_n| \leq \varepsilon.$
6. $\forall \varepsilon \in (0,1),\ \exists n_0 \in \mathbb{N}$ s.t. $\forall n \in \mathbb{N},\quad n \geq n_0 \implies |L - a_n| < \varepsilon.$
7. $\forall \varepsilon > 0,\ \exists n_0 \in \mathbb{N}$ s.t. $\forall n \in \mathbb{N},\quad n \geq n_0 \implies |L - a_n| < \dfrac{1}{\varepsilon}.$
8. $\forall k \in \mathbb{Z}^+ > 0,\ \exists n_0 \in \mathbb{N}$ s.t. $\forall n \in \mathbb{N},\quad n \geq n_0 \implies |L - a_n| < k.$
9. $\forall k \in \mathbb{Z}^+ > 0,\ \exists n_0 \in \mathbb{N}$ s.t. $\forall n \in \mathbb{N},\quad n \geq n_0 \implies |L - a_n| < \dfrac{1}{k}.$

Definition of limit of a sequence (continued)

Let $\{a_n\}_{n=0}^{\infty}$ be a sequence. Let $L \in \mathbb{R}$.
Which statements are equivalent to "$\{a_n\}_{n=0}^{\infty} \longrightarrow L$"?

8. $\forall \varepsilon > 0$, the interval $(L - \varepsilon, L + \varepsilon)$ contains all the elements of the sequence, except the first few.
9. $\forall \varepsilon > 0$, the interval $(L - \varepsilon, L + \varepsilon)$ contains all the elements of the sequence, except finitely many.
10. $\forall \varepsilon > 0$, the interval $(L - \varepsilon, L + \varepsilon)$ contains cofinitely many of the terms of the sequence.
11. $\forall \varepsilon > 0$, the interval $[L - \varepsilon, L + \varepsilon]$ contains cofinitely many of the terms of the sequence.
12. Every interval that contains L must contain cofinitely many of the terms of the sequence.
13. Every open interval that contains L must contain cofinitely many of the terms of the sequence.

Notation: "cofinitely many" = "all but finitely many"

Properties

All the usual properties you know for limits of functions more or less applies to limits of sequences. In particular, (provided the RHS limits exist):

1. $\lim_{n\to\infty}(a_n + kb_n) = \lim_{n\to\infty} a_n + k \lim_{n\to\infty} b_n$.
2. $\lim_{n\to\infty}(a_n b_n) = \lim_{n\to\infty} a_n \lim_{n\to\infty} b_n$.
3. $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$ provided $\lim_{n\to\infty} b_n \neq 0$.
4. If f is continuous, then $\lim_{n\to\infty} f(a_n) = f(\lim_{n\to\infty} a_n)$.
5. If $a_n \leq b_n \leq c_n$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n$, then $\lim_{n\to\infty} b_n$ exists and equal the two limits.

Examples

Construct 8 examples of sequences.
If any of them is impossible, cite a theorem to justify it.

		convergent	divergent
monotonic	bounded	???	???
	unbounded	???	???
not monotonic	bounded	???	???
	unbounded	???	???

Warm up - True or false

1. Every convergent sequence is eventually monotone, that is, eventually increasing or decreasing.

2. If $\lim_{n\to\infty} a_n = L$ then $\lim_{n\to\infty} a_{n^3} = L$.

3. If $\lim_{n\to\infty} a_{2n} = L$ then $\lim_{n\to\infty} a_n = L$.

4. If a sequence diverges and is increasing, then there exists $n \in \mathbb{N}$ such that $a_n > 100$.

Warm up - True or false

1. If a sequence is non-decreasing and non-increasing, then it is convergent.

2. If a sequence is not decreasing and not increasing, then it is convergent.

3. If a sequence is increasing and decreasing, then it is convergent.

Suppse a_n converges and every number in the sequence is an integer. What can you say about the sequence?

A suspicious calculation – What is wrong?

The sequence $\{a_n\}_{n=0}^{\infty}$ defined by

$$\begin{cases} a_0 = 1 \\ \forall n \in N, \quad a_{n+1} = 1 - a_n \end{cases}$$

has limit $1/2$.

Proof.

- Let $L = \lim_{n\to\infty} a_n$.
- $a_{n+1} = 1 - a_n$
- $\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} [1 - a_n]$
- $L = 1 - L$
- $L = 1/2$.

□

Application of MCT

We define the sequence $\{a_n\}_{n=0}^{\infty}$ recursively as follows:

$a_0 = \sqrt{2}$

$a_{n+1} = \sqrt{2 + a_n} \ \forall n \in \mathbb{N}$

Rough work:

1. Guess whether a_n is increasing or decreasing. Don't try to prove it yet.
2. If a_n does converge to some a, taking limits of the recursive relation, what must a be? (**Keep in mind this is completely hypothetical as you have not yet proved that a_n converges.**)
3. Guess an upper bound and a lower bound for a_n using 1 and 2, which is needed for MCT to work?

Proofs:

4. Prove your guess in 1.
5. Prove your bound in 3.
6. Does a_n converge? If so what does it converge to?

Much less than and the Big Theorem

Much less than

Given positive sequences a_n and b_n,
we say $a_n \ll b_n$ iff $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$

The Big Theorem

$$ln(n) \ll n^a \ll c^n \ll n! \ll n^n$$

for every $a > 0, c > 1$.

Calculations

1. $$\lim_{n\to\infty} \frac{n! + 2e^n}{3n! + 4e^n}$$

2. $$\lim_{n\to\infty} \frac{2^n + (2n)^2}{2^{n+1} + n^2}$$

3. $$\lim_{n\to\infty} \frac{5n^5 + 5^n + 5n!}{n^n}$$

Much less than – True or False

Let $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ be positive sequences.

1. IF $a_n \ll b_n$, THEN $\forall m \in \mathbb{N},\ a_m < b_m$.
2. IF $a_n \ll b_n$, THEN $\exists m \in \mathbb{N}$ s.t. $a_m < b_m$.
3. IF $a_n \ll b_n$, THEN $\exists n_0 \in \mathbb{N}$ s.t. $\forall m \in \mathbb{N},\ m \geq n_0 \implies a_m < b_m$.
4. IF $\forall m \in \mathbb{N},\ a_m < b_m$, THEN $a_n \ll b_n$.
5. IF $\exists m \in \mathbb{N}$ s.t. $a_m < b_m$, THEN $a_n \ll b_n$.
6. IF $\exists n_0 \in \mathbb{N}$ s.t. $\forall m \in \mathbb{N},\ m \geq n_0 \implies a_m < b_m$, THEN $a_n \ll b_n$.

Refining the big theorem

1. Construct a sequence $\{u_n\}_n$ such that

$$\begin{cases} \forall a < 2, & n^a \ll u_n \\ \forall a \geq 2, & u_n \ll n^a \end{cases}$$

2. Construct a sequence $\{v_n\}_n$ such that

$$\begin{cases} \forall a \leq 2, & n^a \ll v_n \\ \forall a > 2, & v_n \ll n^a \end{cases}$$