Today's topics and news

- Topic: Rational functions, volumes, sequences.
- **Homework for Friday:** Watch videos 11.3 - 11.8.
- **Test 3** takes place Monday, July 22nd, 6:10 - 8PM in EX100. It will cover PL7 - PL10. This will include everything in today's lecture before sequences. Look at Quercus: Modules: Information on Test 3 for more details.

Rational integrals

1. Calculate $\displaystyle\int \frac{1}{x+a}\,dx$
2. Reduce to common denominator $\displaystyle\frac{2}{x} - \frac{3}{x+3}$
3. Calculate $\displaystyle\int \frac{-x+6}{x^2+3x}\,dx$
4. Calculate $\displaystyle\int \frac{1}{x^2+3x}\,dx$
5. Calculate $\displaystyle\int \frac{1}{x^3-x}\,dx$

Repeated factors

1. Calculate $\displaystyle\int \frac{1}{(x+1)^n}\,dx \quad \text{for } n > 1$
2. Calculate $\displaystyle\int \frac{(x+1)-1}{(x+1)^2}\,dx$
3. Calculate $\displaystyle\int \frac{2x+6}{(x+1)^2}\,dx$
4. Calculate $\displaystyle\int \frac{x^2}{(x+1)^3}\,dx$
5. How would you calculate $\displaystyle\int \frac{\text{polynomial}}{(x+1)^3}\,dx$?

Irreducible quadratics

1. Calculate $\int \frac{1}{x^2+1}\,dx$ and $\int \frac{x}{x^2+1}\,dx$.
   Hint: These two are very short.
2. Calculate $\int \frac{2x+3}{x^2+1}\,dx$
3. Calculate $\int \frac{x^3}{x^2+1}\,dx$
4. Calculate $\int \frac{x}{x^2+x+1}\,dx$

   Hint: Transform it into one like the previous ones

Messier rational functions

1. How could we compute an integral of the form
$$\int \frac{\text{polynomial}}{(x+1)^3(x+2)}dx \ ?$$

2. How could we compute an integral of the form
$$\int \frac{\text{polynomial}}{(x+1)^3(x+2)x^4(x^2+1)(x^2+4x+7)}dx \ ?$$

An equation for volumes by "slicing"

Let $0 < a < b$.
Let f be a continuous, positive function defined on $[a, b]$.
Let R be the region in the first quadrant bounded between the graph of f and the x-axis.

Revolve R around the x-axis, what shape does the line over each x-value become? What is the area of this shape?

Find a formula for the volume of the solid of revolution obtained by rotation the region R around the x-axis.

An equation for volumes by "cylindrical shells"

Let $0 < a < b$.
Let f be a continuous, positive function defined on $[a, b]$.
Let R be the region in the first quadrant bounded between the graph of f and the x-axis.

Revolve R around the y-axis, what shape does the line over each x-value become?What is the area of this shape?

Find a formula for the volume of the solid of revolution obtained by rotation the region R around the y-axis.

Many axis of rotation

Let R be the region in the first quadrant bounded between the curves with equations $y = x^3$ and $y = \sqrt{32x}$. Compute the volume of the solid of revolution obtained by rotating R around...

1. ... the x-axis using both methods
2. ... the line $y = -1$ using either methods
3. ... the y-axis using either methods

Doghnut

Let R be the region inside the curve with equation

$$(x-1)^2 + y^2 = 1.$$

Rotate R around the line with equation $y = 4$. The resulting solid is called a *torus*.

1. Draw a picture and convince yourself that a torus looks like a doughnut.
2. Compute the volume of the torus as an integral using the cylindrical shell method.
3. Compute the volume of the torus as an integral using the slicing method.

Integrating along axis

Fill in which method would apply

	dx	dy
Region in x-y plane rotated about x-axis	?	?
Region in x-y plane rotated about y-axis	?	?

Note: Depending on whether your region is described by functions of x or functions of y, one of the two choices of dx or dy might be better.

Challenge

Two cylinders (of infinite length) have the same radius R and their axes meet at a right angle. Find the volume of their intersection.

Hint: You can slice the resulting solid by parallel cuts in three different directions. One of the three makes the problem much, much simpler than the other two.

Warm up

Write a formula for the general term of these sequences

1. $\{a_n\}_{n=0}^{\infty} = \{\, 1,\ 4,\ 9,\ 16,\ 25,\ \ldots \,\}$
2. $\{b_n\}_{n=1}^{\infty} = \{\, 1,\ -2,\ 4,\ -8,\ 16,\ -32,\ \ldots \,\}$
3. $\{c_n\}_{n=1}^{\infty} = \left\{ \frac{2}{1!}, \frac{3}{2!}, \frac{4}{3!}, \frac{5}{4!}, \ldots \right\}$
4. $\{d_n\}_{n=1}^{\infty} = \{\, 1,\ 4,\ 7,\ 10,\ 13,\ \ldots \,\}$

True or False?

Let f be a function with domain at least $[1, \infty)$.
We define a sequence as $a_n = f(n)$.
Let $L \in \mathbb{R}$.

1. IF $\lim_{x \to \infty} f(x) = L$, THEN $\lim_{n \to \infty} a_n = L$.
2. IF $\lim_{n \to \infty} a_n = L$, THEN $\lim_{x \to \infty} f(x) = L$.
3. IF $\lim_{n \to \infty} a_n = L$, THEN $\lim_{n \to \infty} a_{n+1} = L$.

Definition of limit of a sequence

Let $\{a_n\}_{n=0}^{\infty}$ be a sequence. Let $L \in \mathbb{R}$.
Which statements are equivalent to "$\{a_n\}_{n=0}^{\infty} \longrightarrow L$"?

1. $\forall \varepsilon > 0,\ \exists n_0 \in \mathbb{N} \text{ s.t. } \forall n \in \mathbb{N},\quad n \geq n_0 \implies |L - a_n| < \varepsilon.$
2. $\forall \varepsilon > 0,\ \exists n_0 \in \mathbb{N} \text{ s.t. } \forall n \in \mathbb{N},\quad n > n_0 \implies |L - a_n| < \varepsilon.$
3. $\forall \varepsilon > 0,\ \exists n_0 \in \mathbb{R} \text{ s.t. } \forall n \in \mathbb{N},\quad n \geq n_0 \implies |L - a_n| < \varepsilon.$
4. $\forall \varepsilon > 0,\ \exists n_0 \in \mathbb{N} \text{ s.t. } \forall n \in \mathbb{R},\quad n \geq n_0 \implies |L - a_n| < \varepsilon.$
5. $\forall \varepsilon > 0,\ \exists n_0 \in \mathbb{N} \text{ s.t. } \forall n \in \mathbb{N},\quad n \geq n_0 \implies |L - a_n| \leq \varepsilon.$
6. $\forall \varepsilon \in (0, 1),\ \exists n_0 \in \mathbb{N} \text{ s.t. } \forall n \in \mathbb{N},\quad n \geq n_0 \implies |L - a_n| < \varepsilon.$
7. $\forall \varepsilon > 0,\ \exists n_0 \in \mathbb{N} \text{ s.t. } \forall n \in \mathbb{N},\quad n \geq n_0 \implies |L - a_n| < \frac{1}{\varepsilon}.$
8. $\forall k \in \mathbb{Z}^+ > 0,\ \exists n_0 \in \mathbb{N} \text{ s.t. } \forall n \in \mathbb{N},\quad n \geq n_0 \implies |L - a_n| < k.$
9. $\forall k \in \mathbb{Z}^+ > 0,\ \exists n_0 \in \mathbb{N} \text{ s.t. } \forall n \in \mathbb{N},\quad n \geq n_0 \implies |L - a_n| < \frac{1}{k}.$

Definition of limit of a sequence (continued)

Let $\{a_n\}_{n=0}^{\infty}$ be a sequence. Let $L \in \mathbb{R}$.
Which statements are equivalent to "$\{a_n\}_{n=0}^{\infty} \longrightarrow L$"?

8. $\forall \varepsilon > 0$, the interval $(L - \varepsilon, L + \varepsilon)$ contains all the elements of the sequence, except the first few.
9. $\forall \varepsilon > 0$, the interval $(L - \varepsilon, L + \varepsilon)$ contains all the elements of the sequence, except finitely many.
10. $\forall \varepsilon > 0$, the interval $(L - \varepsilon, L + \varepsilon)$ contains cofinitely many of the terms of the sequence.
11. $\forall \varepsilon > 0$, the interval $[L - \varepsilon, L + \varepsilon]$ contains cofinitely many of the terms of the sequence.
12. Every interval that contains L must contain cofinitely many of the terms of the sequence.
13. Every open interval that contains L must contain cofinitely many of the terms of the sequence.

Notation: "cofinitely many" = "all but finitely many"