Today's topics and news

- Topic: FTC, Integration by substitution
- **Homework for Friday:** Watch videos 9.5 - 9.12, 9.15 - 9.17.
- **PS6** submission links have been sent out. It is due Wednesday, July 17th.
- **Wednesday office hours** will be moved to 5 - 6 to accomodate tutorials. Friday office hours remains the same.

FTC 1

Fundamental Theorem of Calculus 1

Given f is **continuous** on some open interval $\mathbb{I}$ and $a \in \mathbb{I}$,

Define $F(x) = \int_a^x f(t)dt$ $\forall$x$\in \mathbb{I}$,

Then F is **differentiable** and $F'(x) = f(x)$.

This lets us differentiate **accumulation functions**. It tells us that for a continuous function any function defined from its integrals is an antiderivative. This allows us to relate $\int_a^x$ to $\int$.

Counterexample

Give an example where FTC 1 fails if f is only assumed to be integrable.

True or False?

Let f and g be differentiable functions with domain $\mathbb{R}$.
Assume that $f'(x) = g(x)$ for all x.
Which of the following statements must be true?

1. $f(x) = \int_0^x g(t)dt$.
2. If $f(0) = 0$, then $f(x) = \int_0^x g(t)dt$.
3. If $g(0) = 0$, then $f(x) = \int_0^x g(t)dt$.
4. There exists $C \in \mathbb{R}$ such that $f(x) = C + \int_0^x g(t)dt$.
5. There exists $C \in \mathbb{R}$ such that $f(x) = C + \int_1^x g(t)dt$.

Examples of FTC-1

Compute the derivative of the following functions

1. $F_1(x) = \int_0^1 e^{-t^2} dt.$
2. $F_2(x) = \int_0^x e^{-\sin t} dt.$
3. $F_3(x) = \int_1^{x^2} \frac{\sin t}{t^2} dt.$
4. $F_4(x) = \int_x^7 \sin^3\left(\sqrt{t}\right) dt.$
5. $F_5(x) = \int_{2x}^{x^2} \frac{1}{1+t^3} dt.$

An integral equation

Assume f is a continuous function that satisfies, for every $x \in \mathbb{R}$:

$$\int_0^x e^t f(t)dt = \frac{\sin x}{x^2+1}$$

Find an explicit expression for $f(x)$.

An application of FTC-1

Use FTC-1 to prove that, for every $x > 0$

$$\int_0^x \frac{dt}{1+t^2} + \int_0^{1/x} \frac{dt}{1+t^2} = \frac{\pi}{2}$$

What happens for $x < 0$?

FTC 2

FTC 2

Given f integrable over $[a, b]$,

Suppose F is continuous on $[a, b]$ and F'(x) = f(x) on (a, b) (i.e. F is an antiderivative of f on $[a, b]$), then $\int_a^b f(x)dx = F(b) - F(a) = F(x)|_a^b$

FTC 2 relates $\int$ to $\int_a^b$.

FTC 2 turns the task of finding definite integrals into the task of finding antiderivatives and then evaluating on endpoints. The alternative would be to evaluate the limit of Riemann sums which is time-consuming.

FTC 2

FTC 2 (weaker version)

Given f continuous over $[a, b]$,

Suppose F is continuous on $[a, b]$ and F'(x) = f(x) on (a, b) (i.e. F is an antiderivative of f on $[a, b]$), then $\int_a^b f(x)dx = F(b) - F(a) = F(x)|_a^b$

The stronger version is harder to prove. The weaker version is basically a corollary of FTC 1.

What is wrong?

$$\int_{-1}^{1} \frac{1}{x^4} dx \; = \; \frac{-1}{3x^3}\bigg|_{-1}^{1} = \frac{-2}{3}$$

However, x^4 is always positive, so the integral should be positive.

Examples of FTC-2 — Definite integrals

Compute

1. $$\int_1^2 x^3 dx$$
2. $$\int_0^1 \left[e^x + e^{-x} - \cos(2x)\right] dx$$
3. $$\int_{1/2}^{1/\sqrt{2}} \frac{4}{\sqrt{1-x^2}} dx$$
4. $$\int_{\pi/4}^{\pi/3} \sec^2 x \; dx$$
5. $$\int_1^2 \left[\frac{d}{dx}\left(\frac{\sin^2 x}{1+\arctan^2 x + e^{-x^2}}\right)\right] dx$$

Examples of FTC-2 — Areas

Calculate the area of the bounded region...

1. ... between $y = \cos x$, the x-axis, from $x = 0$ to $x = \pi$.

2. ... between $y = x^2 + 3$ and $y = 3x + 1$.

Definite integral via substitution

This final answer is right, but the write-up is WRONG. Why?

Calculate $I = \int_0^2 \sqrt{x^3+1}\, x^2 dx$

Wrong answer

Substitution: $u = x^3 + 1,\ du = 3x^2 dx.$

$$I = \frac{1}{3}\int_0^2 \sqrt{x^3+1}\,(3x^2 dx) = \frac{1}{3}\int_0^2 u^{1/2}\, du$$

$$= \frac{1}{3}\,\frac{2}{3}\ u^{3/2}\Big|_0^2 = \frac{1}{9}\left(x^3+1\right)^{2/3}\Big|_0^2$$

$$= \frac{2}{9}\left(2^3+1\right)^{3/2} - \frac{2}{9}\left(0+1\right)^{3/2} = \frac{52}{9}$$

Practice: integration by substitution

Figure out the substitution that will work for the following:

1. $\displaystyle\int \frac{\sin\sqrt{x}}{\sqrt{x}}dx$
2. $\displaystyle\int e^x cos\left(e^x\right) dx$
3. $\displaystyle\int \cot x \, dx$
4. $\displaystyle\int x^2\sqrt{x+1}\,dx$
5. $\displaystyle\int \frac{e^{2x}}{\sqrt{e^x+1}}\,dx$
6. $\displaystyle\int \frac{(\ln\ln x)^2}{x\ln x}\,dx$
7. $\displaystyle\int xe^{-x^2}\,dx$

Odd functions

Theorem

Let f be a continuous function. Let $a > 0$. IF f is odd, THEN

$$\int_{-a}^{a} f(x)dx = ???$$

1. Draw a picture to interpret the theorem geometrically.
2. Write down the definition of "odd function".
3. Prove the theorem!
   Hint: Write the integral as sum of two pieces. Use a substitution in one of them to show that they cancel with each other.

Warm up

Calculate

$$\int \frac{\sin \sqrt{x}}{\sqrt{x}}\, dx$$

Hint: Use the substitution $u = \sqrt{x}$.