Today's topics and news

- **Topics:** Integrability; Riemann integrals, FTC
- **Homework for Wednesday:** Watch videos 8.3 - 8.7, 9.1 - 9.4.
- **Homework for Friday:** Watch videos 9.5 - 9.12, 9.15 - 9.17.

Joining partitions

Assume

$$L_P(f) = 2, \quad U_P(f) = 6$$
$$L_Q(f) = 3, \quad U_Q(f) = 8$$

1. Is $P \subseteq Q$?
2. Is $Q \subseteq P$?
3. What can you say about $L_{P \cup Q}(f)$ and $U_{P \cup Q}(f)$?

Lower and upper sums

Let f be a **decreasing**, bounded function on $[a, b]$.
Let $P = \{x_0, x_1, \ldots, x_N\}$ be a partition of $[a, b]$

Which one (or ones) is a valid equation for $L_P(f)$? For $U_f(P)$?

1. $\sum_{i=0}^{N} \Delta x_i \, f(x_i)$
2. $\sum_{i=1}^{N} \Delta x_i \, f(x_i)$
3. $\sum_{i=0}^{N-1} \Delta x_i \, f(x_i)$
4. $\sum_{i=1}^{N} \Delta x_i \, f(x_{i+1})$
5. $\sum_{i=1}^{N} \Delta x_i \, f(x_{i-1})$
6. $\sum_{i=0}^{N-1} \Delta x_{i+1} \, f(x_i)$

Recall: $\Delta x_i = x_i - x_{i-1}$.

Example 1: a constant function

Consider the function $f(x) = 2$ on $[0, 4]$.

1. Given $P = \{0, 1, e, \pi, 4\}$, compute $L_P(f)$ and $U_P(f)$.
2. Explicitly compute *all* the upper sums and *all* the lower sums.
3. Compute $\underline{I_0^4}(f)$
4. Compute $\overline{I_0^4}(f)$
5. Is f integrable on $[0, 4]$?

Example 2: a non-continuous function

Consider the function $f(x) = \begin{cases} 0 & x = 0 \\ 5 & 0 < x \leq 1 \end{cases}$, defined on $[0,1]$.

1. Let $P = \{0, 0.2, 0.5, 0.9, 1\}$.
   Calculate $L_P(f)$ and $U_P(f)$ for this partition.
2. Fix an arbitrary partition $P = \{x_0, x_1, \ldots, x_N\}$ of $[0,1]$.
   What is $U_P(f)$? What is $L_P(f)$? (Draw a picture!)
3. Find a partition P such that $L_P(f) = 4.99$.
4. What is the upper integral, $\overline{I_0^1}(f)$?
5. What is the lower integral, $\underline{I_0^1}(f)$?
6. Is f integrable on $[0,1]$?

Is this possible?

Find bounded functions f and g on $[0, 1]$ such that

- f is not integrable on $[0, 1]$,
- g is not integrable on $[0, 1]$,
- $f + g$ is integrable on $[0, 1]$.

or prove this is impossible.

Properties of the integral

Assume we know the following

$$\int_0^2 f(x)dx = 3, \quad \int_0^4 f(x)dx = 9, \quad \int_0^4 g(x)dx = 2.$$

Compute:

1. $\int_0^2 f(t)dt$
2. $\int_0^2 f(t)dx$
3. $\int_2^0 f(x)dx$
4. $\int_2^4 f(x)dx$
5. $\int_{-2}^0 f(x)dx$
6. $\int_0^4 [f(x) - 2g(x)] \; dx$

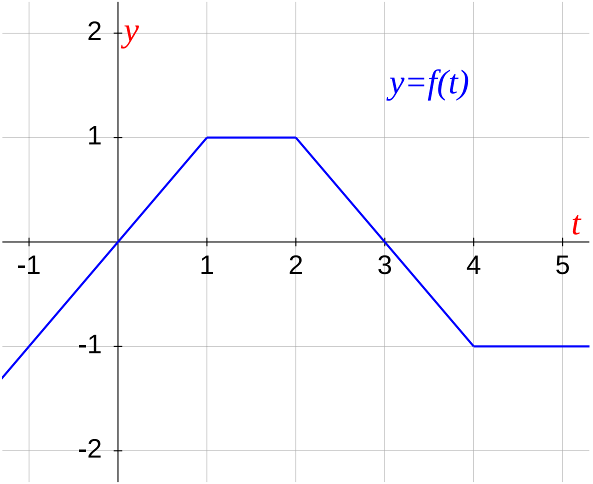
Riemann sums backwards

Interpret the following limits as integrals:

1. $$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n} \sin \frac{i}{n}$$

2. $$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{n+i}{n^2}$$

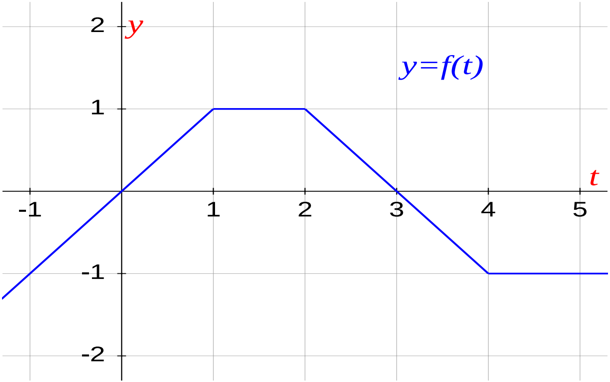
Towards FTC



Compute:

1. $\int_0^1 f(t)dt$
2. $\int_0^2 f(t)dt$
3. $\int_0^3 f(t)dt$
4. $\int_0^4 f(t)dt$
5. $\int_0^5 f(t)dt$

Towards FTC (continued)



Call $F(x) = \int_0^x f(t)dt$. This is a new function.

- Sketch the graph of $y = F(x)$.
- Using the graph you just sketched, sketch the graph of $y = F'(x)$.

Three expressions containing the integral symbol

Fix a, $b \in \mathbb{R}$. Given a function f integrable on $\mathbb{R}$.

Recall what type of object the following are and how they are defined.

1. $\int_a^b f(x)dx$
2. $\int_a^x f(t)dt$
3. $\int f(x)dx$

Which of the three are related from the definition?

Functions defined by integrals

Which ones of these are valid ways to define functions?

1. $F(x) = \int_0^x \frac{t}{1+t^8}\,dt$
2. $F(x) = \int_0^x \frac{x}{1+x^8}\,dx$
3. $F(x) = \int_0^x \frac{x}{1+t^8}\,dt$
4. $F(x) = \int_0^{x^2} \frac{t}{1+t^8}\,dt$
5. $F(x) = \int_{\sin x}^{e^x} \frac{t}{1+t^8}\,dt$
6. $F(x) = \int_0^3 \frac{t}{1+x^2+t^8}\,dt$
7. $F(x) = x\int_{\sin x}^{e^x} \frac{t}{1+x^2+t^8}\,dt$
8. $F(x) = t\int_{\sin x}^{e^x} \frac{t}{1+x^2+t^8}\,dt$

The "ε–reformulation" of integrability - First proof

You are going to prove

Claim

Let f be a bounded function on $[a, b]$.

- IF f is integrable on $[a, b]$
- THEN "$\forall \varepsilon > 0$, $\exists$ a partition P of $[a, b]$, s.t. $U_P(f) - L_P(f) < \varepsilon$".

1. Recall the definition of "f is integrable on $[a, b]$".
2. Write down the structure of the proof.
3. Fix $\varepsilon > 0$. Show there is a partition P s.t. $U_P(f) < \overline{I_a^b}(f) + \frac{\varepsilon}{2}$. Why?
4. Assume f is integrable on $[a, b]$. Fix $\varepsilon > 0$.
   Show there are partitions P_1 and P_2 s.t. $U_{P_1}(f) - L_{P_2}(f) < \varepsilon$.
5. Using P_1 and P_2 from the previous step, construct a partition P such that $U_P(f) - L_P(f) < \varepsilon$.
6. Write down a proof for the claim.

The "ε–reformulation" of integrability - Second proof

You are going to prove

Claim

Let f be a bounded function on $[a, b]$.

- IF "$\forall \varepsilon > 0$, $\exists$ a partition P of $[a, b]$, s.t. $U_P(f) - L_P(f) < \varepsilon$".
- THEN f is integrable on $[a, b]$

1. Recall the definition of "f is integrable on $[a, b]$".
2. For any partition P, order the quantities $U_P(f)$, $L_P(f)$, $\overline{I_a^b}(f)$, $\underline{I_a^b}(f)$.
3. For any partition P, order the quantities $U_P(f) - L_P(f)$, $\overline{I_a^b}(f) - \underline{I_a^b}(f)$, and 0.
4. Write down a proof for the claim.

The "ε–reformulation" of integrability

The "ε–reformulation" of integrability

Let f be a bounded function on $[a, b]$.

f is integrable on $[a, b]$

IFF

$\forall \varepsilon > 0$, $\exists$ a partition P of $[a, b]$, s.t. $U_P(f) - L_P(f) < \varepsilon$.