Introduction

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Today's topics and news

- **Topics:** Sigma and sums; Sup and inf; Definition of the integral

- **Homework for Friday:** Watch videos 7.8 - 7.12, 8.1 and 8.2.

- **PS5** is due Wednesday, July 10th. You should have already received Crowdmark submission links. Contact me if you have not.

Sigma

Recall (7.2) that $\sum$ is called sigma and is a notation used the denote sum.

Given $a_1, a_2, ...a_n \in \mathbb{R}$, $\sum_{k=1}^{n} a_k = a_1 + a_2 + ... + a_n$.

Here k is a dummy variable and has no meaning outside of $\sum$.

Define $\forall k \in \mathbb{N}, a_k = 2k + 1$. Compute:

1. $\sum_{k=2}^{4} a_k$.
2. $\sum_{i=2}^{4} a_k$.
3. $\sum_{i=2}^{4} a_i$.

Write these sums with sigma notation

1. $1^5 + 2^5 + 3^5 + 4^5 + \ldots + 100^5$

2. $\frac{2}{4^2} + \frac{2}{5^2} + \frac{2}{6^2} + \frac{2}{7^2} + \ldots + \frac{2}{N^2}$

3. $\frac{1}{1!} - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \ldots + \frac{1}{81!}$

4. $\frac{x^2}{3!} + \frac{2x^3}{4!} + \frac{3x^4}{5!} + \frac{4x^5}{6!} + \ldots + \frac{999x^{1000}}{1001!}$

Re-writing sums

1. $$\sum_{i=1}^{100} \tan i - \sum_{i=1}^{50} \tan i = \sum_{???}^{???} ???$$

2. $$\sum_{i=1}^{N} (2i-1)^5 = \sum_{i=0}^{N-1} ???$$

Double sums

Compute:

1. $\sum_{i=1}^{n}(\sum_{k=1}^{n} 1)$
2. $\sum_{i=1}^{n}(\sum_{k=1}^{i} 1)$
3. $\sum_{i=1}^{n}(\sum_{k=1}^{i} i)$
4. $\sum_{i=1}^{n}(\sum_{k=1}^{i} k)$
5. $\sum_{i=1}^{n}(\sum_{k=1}^{i} (ik))$

Use the following formulas:

1. $\sum_{k=1}^{n} k = \frac{(n)(n+1)}{2}$
2. $\sum_{k=1}^{n} k^2 = \frac{(n)(n+1)(2n+1)}{6}$
3. $\sum_{k=1}^{n} k^3 = \frac{(n)^2(n+1)^2}{4}$

Supremum and infimum

Given $A \subseteq \mathbb{R}$. Recall $\sup(A)$ is by definition the least upperbound of A provided such a number exists.

Therefore, to check if a number a is in fact the supremum of a given set A, one needs to check two conditions:

1. a is an upperbound of A (i.e. $\forall x \in A, a \geq x$).

2. if b is an upperbound of A, then $a \leq b$ (i.e. $\forall b \in \mathbb{R}$, if $(\forall x \in A, b \geq x)$, then $a \leq b$).

Empty set

1. Does $\emptyset$ have an upper bound ?
2. Does $\emptyset$ have a supremum?
3. Does $\emptyset$ have a maximum?
4. Is $\emptyset$ bounded above?

Recall:

Let $A \subseteq \mathbb{R}$. Let $a \in \mathbb{R}$.

- a is an **upper bound** of A means $\forall x \in A, x \leq a$.
- a is the **least upper bound** (lub) or **supremum** (sup) of A means
  - a is an upper bound of A, and
  - there are no smaller upper bounds.

sup and inf existence theorem

sup existence theorem

Let $A \subseteq \mathbb{R}$,
A has a supremum iff A is bounded above and non-empty.

Equivalent definitions of supremum

Assume u is an upper bound of the set A, which of the following statements are equivalent to $u = \sup(A)$?

1. $\forall v \leq u$, v is not an upper bound of A.
2. $\forall v < u$, v is not an upper bound of A.
3. $\forall v < u$, $\exists x \in A$ s.t. $v < x$.
4. $\forall v < u$, $\exists x \in A$ s.t. $v \leq x$.
5. $\forall v < u$, $\exists x \in A$ s.t. $v < x \leq u$.
6. $\forall v < u$, $\exists x \in A$ s.t. $v < x < u$.
7. $\forall \epsilon > 0$, $\exists x \in A$ s.t. $u - \epsilon < x \leq u$.
8. $\forall \epsilon > 0$, $\exists x \in A$ s.t. $u - \epsilon < x < u$.

Sup and inf proof

Let $A = [0, 1)$

Prove $\inf(A) = 0$:
1. Check 0 a lower bound.
2. Suppose l is another lower bound of A, why does 0 have to be larger?

Prove $\sup(A) = 1$:
1. Check 1 an upper bound.
2. Suppose u is another upper bound of A, and assume it's less than 1, come up with a number in A (using u) which is for sure larger than u, therefore contradicting the fact that u is an upper bound.

Partitions

Which of the following are partitions of $[0, 2]$?

1. $[0, 2]$
2. $(0, 2)$
3. $\{0, 2\}$
4. $\{1, 2\}$
5. $\{0, 1, 1.5, 2\}$

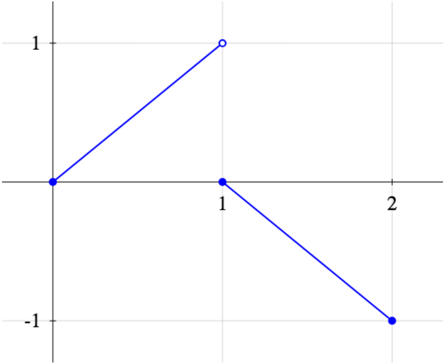
A **partition** of $[a, b]$ is **expressed** as a finite set S where $S \subseteq [a, b]$ and $a,\ b \in S$. It should be **thought of** as a way of dividing up the interval $[a, b]$, where you divide $[a, b]$ at all elements of S. Partitions are often written in order.

Definition of upper sum

Given a **bounded function** f on $[a, b]$ **and** a **partition** P, there are two ways to estimate the"signed area under the curve of f". These are called upper sum $U_P(f)$ and the lower sum $L_P(f)$. As you will see in the next slide, these estimates **depend** on the partition.

Exercise: Given a partition $\{a = x_0 < x_1 < ...x_n = b\}$ of $[a, b]$ and a bounded function f. Define $U_P(f)$.

Computing $U_P(f)$



Compute $U_P(f)$ for the following partitions:

1. $\{0, 2\}$
2. $\{0, 0.5, 1.5, 2\}$

Upper and lower integrals

We see that given a bounded function f on $[a, b]$ and a partition P of $[a, b]$, we can produce two estimates for the area under f between a and b, one of which is an underestimate and one of which is an overestimate.

There are infinitely many partitions, each giving their own (potentially) distinct over- and underestimates. If we want to get a true notion of the area, we can look all possible partitions and their corresponding overestimates $U_P(f)$, and find the "smallest" of all of them. We can similarly look at the all possible partitions and their corresponding underestimates $L_P(f)$, and find the "largest" of all of them. This motivates the following definitions:

1. The upper integral $\overline{I_a^b}(f) :=$
2. The lower integral $\underline{I_a^b}(f) :=$

Integrability

1. The upper integral $\overline{I_a^b}(f) := \inf(\{U_P(f) : \text{P is a partition of } [a,b]\})$
2. The lower integral $\underline{I_a^b}(f) := \sup(\{L_P(f) : \text{P is a partition of [a,b]}\})$

Note there is always a relationship between these two numbers: $\overline{I_a^b}(f) \geq \underline{I_a^b}(f)$.

These are the best possible candidates for area under f and there's no preference for one over the other. That's why if they are not equal (i.e. $\overline{I_a^b}(f) > \underline{I_a^b}(f)$), we don't have a good notion of area and we say the function is not integrable. And if they are equal, then the function is **integrable** and

$$\int_a^b f(x)dx = \overline{I_a^b}(f) = \underline{I_a^b}(f).$$

Trick question

Is $\frac{1}{\sqrt{x}}$ integrable on $[0, 1]$?

Answer: No. It's not bounded on $[0, 1]$ so the theory of integration we have developed does not apply. We will learn this function does have an improper integral in several weeks.