Today's topics and news

- Topic: Taylor series and applications
- **Homework:** Study for the exam!

Lagrange's Remainder Theorem

Lagrange's Remainder Theorem

Suppose f is C^{n+1} on some interval $\mathbb{I}$ containing a.

Let P_n be the n^{th} Taylor Polynomial of f at a.

Consider $R_n(x) = f(x) - P_n(x)$ the remainder,

then for any $x \in \mathbb{I}$, there exists ξ between a and x s.t.

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}$$

Notice that ξ depends on n and x.

Proving a function is analytic

We will now show that $\sin(x)$ is analytic on $\mathbb{R}$. To show a function f is analytic on some D we need to show that f is analytic on each point $a \in \mathbb{D}$. This in turn means that the Taylor series of f centred at a converges to f in a small neighbourhood of a.

We will call $S_a(x)$ the Taylor series for $\sin(x)$ centred around a. We will call $P_{n,a}(x)$ the n^{th} Taylor polynomial for $\sin(x)$ centred around a.

1. We will show that for any $a, x \in \mathbb{R}$, $S_a(x)$ converges to $\sin(x)$. In other words, the small neighbourhood around each a for which S_a converges to f can in fact be taken to be all of $\mathbb{R}$. Use Lagrange's Theorem to write down an expression for the remainder $R_{n,a}(x) = f(x) - P_{n,a}(x)$

2. Show that $\lim_{n \to \infty} R_{n,a}(x) = 0$.

This of course means $\sin(x)$ is analytic on $\mathbb{R}$.

Warm-up: Taylor series gymnastics

Write the following functions in a way where you can easily find their Maclaurin series using series you already know.

1. $f(x) = \dfrac{x^2}{1+x}$
2. $f(x) = (e^x)^2$
3. $f(x) = \sin\left(2x^3\right)$
4. $f(x) = \cos^2 x$
5. $f(x) = \ln \dfrac{1+x}{1-x}$
6. $f(x) = \dfrac{1}{(1+x^2)(1+x)}$

Note: You do not need to take any derivatives. You can reduce them all to other Maclaurin series you know.

Arctan

1. Write the function

$$f(x) = \arctan x$$

as a power series centered at 0.
Hint: Compute the first derivative. Then stop to think.

2. What is $f^{(2019)}(0)$?

Maclaurin series and derivatives at 0

Let f be an analytic function defined on some interval centred at 0. We define a new function g via the equation $g(x) = f(x^2)$.

Find $g^{(n)}(0)$ in terms of the derivatives of f at 0.

Hint: Write a Maclaurin series for g in two different ways.

Review: what's wrong with the following computation?

Let us find the Taylor series for $\cos(x)$ with the Taylor series for $\sin(x)$. We know $\cos(x) = \int \sin(x)dx$ so:

$$\begin{aligned}\cos(x) &= \int \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} dx \\ &= \sum_{n=0}^{\infty} \int \frac{(-1)^n x^{2n+1}}{(2n+1)!} dx \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{(2n+2)!}\end{aligned}$$

Fix this computation.

Add these series

1. $$\sum_{n=2}^{\infty} \frac{(-2)^n}{(2n+1)!}$$

2. $$\sum_{n=0}^{\infty} (4n+1)x^{4n+2}$$

3. $$\sum_{n=0}^{\infty} \frac{2^n}{(2n)!}$$

4. $$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!(n+1)}$$

Add these series

1. $$\sum_{n=2}^{\infty} \frac{(-2)^n}{(2n+1)!}$$ *Hint:* Think of sin

2. $$\sum_{n=0}^{\infty} (4n+1)x^{4n+2}$$ *Hint:* $\frac{d}{dx}\left[x^{4n+1}\right] = ???$

3. $$\sum_{n=0}^{\infty} \frac{2^n}{(2n)!}$$ *Hint:* Write the first few terms. Combine e^x and e^{-x}

4. $$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!(n+1)}$$ *Hint:* Integrate

Limits

Compute these limits by writing out the first few terms of the Maclaurin series of numerator and denominator:

1. $$\lim_{x\to 0} \frac{\sin x - x}{x^3}$$
2. $$\lim_{x\to 0} \frac{6\sin x - 6x + x^3}{x^5}$$
3. $$\lim_{x\to 0} \frac{\cos(x) - 1 + \frac{1}{2}x\sin(x)}{\ln(1+x)^4}$$