Today's topics and news

- Topic: Power series, Taylor series
- **Homework for Friday:** Watch videos 14.11 - 14.14

Approximating functions

Which one of the following functions is a better approximation for $F(x) = e^x$ near 0?

1. $f(x) = 1 + x + \dfrac{x^2}{2}$
2. $g(x) = \sin x + \cos x + x^2$
3. $h(x) = e^{-x} + 2x$

An explicit equation for Taylor polynomials

1. Find a polynomial P of degree 3 that satisfies

$$P(0) = 1, \quad P'(0) = 5, \quad P''(0) = 3, \quad P'''(0) = -7$$

2. Find *all* polynomials P that satisfy

$$P(0) = 1, \quad P'(0) = 5, \quad P''(0) = 3, \quad P'''(0) = -7$$

3. Find a polynomial P of smallest possible degree that satisfies

$$P(0) = A, \quad P'(0) = B, \quad P''(0) = C, \quad P'''(0) = D$$

4. Find an explicit formula for the 3-rd Taylor polynomial for a C^3 function f at 0.
5. Find an explicit formula for the n-th Taylor polynomial for a C^n function f at 0.

Interval of convergence

Write the Maclaurin series for the following functions

$$f(x) = e^x, \qquad g(x) = \sin x, \qquad h(x) = \cos x$$

Compute the interval of convergence of each of these three series.

Taylor series not at 0

Write the Taylor series...

1. for $f(x) = e^x$ at $a = 2$
2. for $g(x) = \sin x$ at $a = \frac{\pi}{4}$
3. for $H(x) = \frac{1}{x}$ at $a = 3$

You can do these problems in two ways:

- Method 1: Use the substitution $u = x - a$ and reduce it to an old problem (without computing any derivative).
- Method 2: Compute the first few derivatives, guess the pattern (and prove it by induction).

Taylor polynomial of a polynomial

1. Let $f(x) = x^3 + x^2$
   Write the 2nd Taylor polynomial P_2 for f at 0
2. Write the Taylor series for f at 0.
3. Write the 2nd Taylor polynomial for f at 1.
4. Write the 3rd Taylor polynomial for f at 1.

Taylor Series

Taylor Series: For C^∞ functions

Let $a \in \mathbb{R}$. For C^∞ functions $f(x)$,
$S(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$ is the Taylor series of f centred a.

One of the reasons why we want to look at the Taylor series is in the hope that it would nicely approximate the function $f(x)$ near a. This is not always the case. We say

Analyticity

A function f is analytic at a iff,
$S_a(x)$, the Taylor series expansion of f around a converges to $f(x)$ in a neighbourhood of a.
A function f is analytic on some set D iff $\forall a \in D$, it's analytic at a .

Example: We have shown that $f(x) = \frac{1}{1-x}$ is analytic at 0.

A pathological example

Consider the function $f(x)$ defined as follows:

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ e^{\frac{-1}{x^2}} & \text{if } x \neq 0 \end{cases}$$

One can show that this function is C^∞ and $f^{(k)}(0) = 0$ $\forall k \in \mathbb{N}$.

1. Does this function have a Taylor polynomial of order 9 around 0?
2. Does this function have a Taylor Series around 0?
3. Is this function analytic?

Taylor series centred at $x = 1$

Write the Taylor series for $\frac{1}{1-x}$ centred around 0.5.

What is the interval of convergence of this Taylor series? Does it converge to $\frac{1}{1-x}$ on that interval?

Is $\frac{1}{1-x}$ analytic at 0.5?

Lagrange's Remainder Theorem

Suppose f is C^{n+1} on some interval $\mathbb{I}$ containing a.

Let P_n be the n^{th} Taylor Polynomial of f at a.

Consider $R_n(x) = f(x) - P_n(x)$ the remainder,

then for any $x \in \mathbb{I}$, there exists ξ between a and x s.t.

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}$$

Notice that ξ depends on n and x.

Proving a function is analytic

We will now show that $\sin(x)$ is analytic on $\mathbb{R}$. To show a function f is analytic on some D we need to show that f is analytic on each point $a \in \mathbb{D}$. This in turn means that the Taylor series of f centred at a converges to f in a small neighbourhood of a.

We will call $S_a(x)$ the Taylor series for $\sin(x)$ centred around a. We will call $P_{n,a}(x)$ the n^{th} Taylor polynomial for $\sin(x)$ centred around a.

1. We will show that for any $a, x \in \mathbb{R}$, $S_a(x)$ converges to $\sin(x)$. In other words, the small neighbourhood around each a for which S_a converges to f can in fact be taken to be all of $\mathbb{R}$. Use Lagrange's Theorem to write down an expression for the remainder $R_{n,a}(x) = f(x) - P_{n,a}(x)$

2. Show that $\lim\limits_{n\to\infty} R_{n,a}(x) = 0$.

This of course means $\sin(x)$ is analytic on $\mathbb{R}$.

Taylor series gymnastics

Write the following functions in a way where you can easily find their Maclaurin series using series you already know.

1. $f(x) = \dfrac{x^2}{1+x}$
2. $f(x) = (e^x)^2$
3. $f(x) = \sin\left(2x^3\right)$
4. $f(x) = \cos^2 x$
5. $f(x) = \ln \dfrac{1+x}{1-x}$
6. $f(x) = \dfrac{1}{(1+x^2)(1+x)}$

Note: You do not need to take any derivatives. You can reduce them all to other Maclaurin series you know.

Arctan

1. Write the function

$$f(x) = \arctan x$$

as a power series centered at 0.
Hint: Compute the first derivative. Then stop to think.

2. What is $f^{(2019)}(0)$?