Today's topics and news

- Topic: Absolute convergence, ratio test, power series
- **Homework for Wednesday:** Watch videos 14.5 - 14.10
- **Homework for Friday:** Watch videos 14.11 - 14.14

Positive and negative terms

- Let $\sum a_n$ be a series.
- Call $\sum$(P.T.) the sum of only the positive terms of the same series.
- Call $\sum$(N.T.) the sum of only the negative terms of the same series.

IF $\sum$(P.T.) is...	AND $\sum$(N.T.) is...	THEN $\sum a_n$ may be...
CONV	CONV	
∞	CONV	
CONV	$-\infty$	
∞	$-\infty$	

Positive and negative terms

- Let $\sum a_n$ be a series.
- Call $\sum$(P.T.) the sum of only the positive terms of the same series.
- Call $\sum$(N.T.) the sum of only the negative terms of the same series.

	$\sum$(P.T.) may be...	$\sum$(N.T.) may be...
In general		
If $\sum a_n$ is CONV		
If $\sum \lvert a_n \rvert$ is CONV		
If $\sum a_n$ is ABS CONV		
If $\sum a_n$ is COND CONV		
If $\sum a_n = \infty$		
If $\sum a_n$ is DIV (not to ∞ or $-\infty$)		

The inconclusive case of the ratio test

Show the ratio test is inconclusive when $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$.

Ratio test: Convergent or divergent?

Use Ratio test to decide which series are convergent:

1. $$\sum_{n=1}^{\infty} \frac{3^n}{n!}$$
2. $$\sum_{n=1}^{\infty} \frac{(2n)!}{n!^2\, 3^{n+1}}$$
3. $$\sum_{n=2}^{\infty} \frac{1}{\ln n}$$
4. $$\sum_{n=2}^{\infty} \frac{n!}{n^n}$$

Root test

Here is a new convergence test.

Theorem

Let $\sum_n a_n$ be a series. Assume the limit $L = \lim_{n\to\infty} \sqrt[n]{|a_n|}$ exists.

- IF $0 \leq L < 1$ THEN the series is ???
- IF $L > 1$ THEN the series is ???

Without writing an actual proof, guess the conclusion of the theorem and argue why it makes sense.

Hint: Imitate the explanation on Video 13.18 for the Ration Test.
For large values of n, what can you compare $|a_n|$ to? Think about the simplest series $\sum_n b_n$ s.t. $\lim_{n\to\infty} |b_n|^{\frac{1}{n}} = L$.

Power Series

A power series is a series (dependent on some variable, ex. x) of the form

$$\sum_{n=0}^{\infty} c_n(x-a)^n$$

In this form, we say the power series is centred at a.

A power series isn't actually a series because the terms aren't numbers. However, if I plug in a specific number for x, then it becomes a genuine series. Notice a power series always converges to c_0 if I plug in $x = a$.

Depending on what x you plug in, the power series might be convergent or divergent. On the x-values where the power series converges, you can think of the power series as representing some function $f(x)$.

Example: Consider the geometric power series $\sum_{n=0}^{\infty} x^n$. It converges iff $|x| < 1$. On this interval, the power series as a function is equal to the the function $\frac{1}{1-x}$.

Interval of convergence

Find the interval of convergence of each power series:

1. $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$
2. $\displaystyle\sum_{n=1}^{\infty} \frac{(x-5)^n}{n^2\, 2^{2n+1}}$
3. $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{42^n} x^n$
4. (Hard!) $\displaystyle\sum_{n=0}^{\infty} \frac{(3n)!}{n!(2n)!}\, x^n$

What can you conclude?

Think of the power series $\sum_n a_n x^n$. Do not assume $a_n \geq 0$.

In each case, may the given series be absolutely convergent (AC)? conditionally convergent (CC)? divergent (D)? all of them?

IF	$\sum_n a_n 3^n$ is ...	AC	CC	D
THEN	$\sum_n a_n 2^n$ may be ...	???	???	???
	$\sum_n a_n (-3)^n$ may be ...	???	???	???
	$\sum_n a_n 4^n$ may be ...	???	???	???

Writing functions as power series

Using the geometric series, we know how to write the function $F(x) = \dfrac{1}{1-x}$ as a power series centered at 0:

$$F(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \text{for } |x| < 1$$

Write these functions as power series centered at 0:

1. $f(x) = \dfrac{1}{1+x}$
2. $g(x) = \dfrac{1}{1-x^2}$
3. $h(x) = \dfrac{1}{2-x}$
4. $G(x) = \ln(1+x)$

The definitions of Taylor polynomial

Let f be a function defined at and near $a \in \mathbb{R}$. Let $n \in \mathbb{N}$.
Let P_n be the n-th Taylor polynomial for f at a.
Which ones of these is true?

1. P_n is an approximation for f of order n near a.
2. f is an approximation for P_n of order n near a.
3. $\lim_{x \to a} [f(x) - P_n(x)] = 0$
4. $\lim_{x \to a} \frac{f(x) - P_n(x)}{(x-a)^n} = 0$
5. $\exists$ a function R_n s.t. $f(x) = P_n(x) + R_n(x)$ and $\lim_{x \to a} \frac{R_n(x)}{(x-a)^n} = 0$
6. $f^{(n)}(a) = P_n^{(n)}(a)$
7. $\forall k = 0, 1, 2, \ldots, n, \quad f^{(k)}(a) = P_n^{(k)}(a)$
8. If x is close to a, then $f(x) = P_n(x)$.

Approximating functions

Which one of the following functions is a better approximation for $F(x) = e^x$ near 0?

1. $f(x) = 1 + x + \frac{x^2}{2}$
2. $g(x) = \sin x + \cos x + x^2$
3. $h(x) = e^{-x} + 2x$

An explicit equation for Taylor polynomials

1. Find a polynomial P of degree 3 that satisfies

$$P(0) = 1, \quad P'(0) = 5, \quad P''(0) = 3, \quad P'''(0) = -7$$

2. Find *all* polynomials P that satisfy

$$P(0) = 1, \quad P'(0) = 5, \quad P''(0) = 3, \quad P'''(0) = -7$$

3. Find a polynomial P of smallest possible degree that satisfies

$$P(0) = A, \quad P'(0) = B, \quad P''(0) = C, \quad P'''(0) = D$$

4. Find an explicit formula for the 3-rd Taylor polynomial for a function f at 0.
5. Find an explicit formula for the n-th Taylor polynomial for a function f at 0.