Today's topics and news

- Topics: Taylor Series, analyticity

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- Homework: Watch videos 14.11 - 14.15

Examples

1. Give a power series which has a radius of convergence of ∞.
2. Give a power series which has a radius of convergence of 0.

Taylor Polynomials

Taylor Polynomial: Most general definition: For C^0 (i.e continuous) functions

Let $a \in \mathbb{R}$. For functions $f(x)$ and $n \in \mathbb{N}$, we say $P(x)$ is the n-th degree Taylor polynomial if P is the (unique) polynomial of smallest possible degree which is an approximtion for f near a of order n.

Taylor Polynomial: For C^n functions

Let $a \in \mathbb{R}$. For C^n functions $f(x)$ and $n \in \mathbb{N}$, we say $P(x)$ is the n-th degree Taylor polynomial if P is the (unique) polynomial of smallest possible degree s.t. $P^{(k)}(a) = f^{(k)}(a)$ for $k = 0, 1, ...n$.

Taylor Polynomials

Given a functon $f(x)$ which is C^k near a, it is actually quite straightforward to write down the polynomial of the smallest degree which satisfies $P^{(k)}(a) = f^{(k)}(a)\ \forall k \leq n$.

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Taylor Polynomial: Most practical definition: For C^n functions

Let $a \in \mathbb{R}$. For C^n functions $f(x)$ and $n \in \mathbb{N}$,

$P(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$ is the n-th degree Taylor polynomial of f near a.

Taylor Series

Taylor Series: For C^∞ functions

Let $a \in \mathbb{R}$. For C^∞ functions $f(x)$,
$S(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$ is the Taylor series of f centred a.

One of the reasons why we want to look at the Taylor series is the hopes that it would nicely approximate the function $f(x)$ near a. This is not always the case. We say

Analyticity

A function f is analytic at a iff,
$S_a(x)$, the Taylor series expansion of f around a converges to $f(x)$ in a neighbourhood of a.
A function f is analytic on some set D if it's analytic at a $\forall a \in D$.

Example: We have shown that $f(x) = \frac{1}{1-x}$ is analytic at 0.

The 4 Taylor series

Write down the Maclaurin series for:

1. $\frac{1}{1-x}$
2. $\sin(x)$
3. $\cos(x)$
4. e^x
5. $f(x) = -1 + (x-1)^2$

Compute their interval of convergence if you do not know them.

Taylor series at different centres

Write down the Taylor series for:

1. e^x centred at 2.
2. $\sin(x)$ centred at $\frac{\pi}{4}$.
3. $\frac{1}{x}$ at 3.

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Hint: There are two methods. You can try to find a pattern in all the derivatives. You can also try to write x (which is centred at 0) in a way which is centred around a: $x = (x - a) + a$.

A pathological example

Consider the function $f(x)$ defined as follows:

$f(x) = e^{\frac{-1}{x^2}}$ if $x \neq 0$
$f(x) = 0$ if $x = 0$.

One can show that this function is C^∞ and $f^{(k)}(0) = 0 \; \forall k \in \mathbb{N}$.

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1. Does this function have a Taylor polynomial of order 9 around 0?
2. Does this function have a Taylor Series around 0?
3. Is this function anayltic?

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1. Does this function have a Taylor polynomial of order 9 around 0? Yes.
2. Does this function have a Taylor Series around 0? Yes.
3. Is this function anayltic? No.

Lagrange's Remainder Theorem

Lagrange's Remainder Theorem

Suppose f is C^{n+1} on some interval $\mathbb{I}$ containing a. Let P_n be the n^{th} Taylor Polynomial of f at a. Consider $R_n(x) = f(x) - P_n(x)$ the remainder,
then for any $x \in \mathbb{I}$, there exists c between a and x s.t.

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

Notice that c depends on n and x.

Approximate $e^{\frac{1}{2}}$ by a rational number so that the error is less than 0.001.

Hint: Apply Lagrange's remainder theorem to P_n the Taylor polynomials centred around 0 and try to bound the remainder term $R_n(\frac{1}{2})$. The bound you get will depend on n. Choose an n so that the remainder is less than 0.001.

Proving a function is analytic

We will now show that $\sin(x)$ is analytic on $\mathbb{R}$. Remember, to show a function f is analytic on some D we need to show that f is analytic on each point $a \in \mathbb{D}$. This in turn means that the Taylor series of f centred at a converges to f in a small neighbourhood of a.

We will call $S_a(x)$ the Taylor series for $\sin(x)$ centred around a. We will call $S_{n,a}(x)$ the n^{th} Taylor polynomial for $\sin(x)$ centred around a.

1. We will show that for any $a, x \in \mathbb{R}$, $S_a(x)$ converges to $\sin(x)$. In other words, the small neighbourhood around each a for which S_a converges to f can in fact be taken to be all of $\mathbb{R}$. Use Lagrange's Theorem to write down an expression for the remainder $R_{a,n}(x) = f(x) - S_{a,n}(x)$

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2. Show that $\lim\limits_{n\to\infty} R_{a,n}(x) = 0$.

This of course means $\sin(x)$ is analytic on $\mathbb{R}$.

Analytic functions

We know the following functions are analytic:

1. e^x on $\mathbb{R}$
2. $\sin(x)$ on $\mathbb{R}$
3. $\cos(x)$ on $\mathbb{R}$
4. $\frac{1}{1-x}$ on $\mathbb{R}\setminus\{1\}$, and therefore $\frac{1}{x}$ on $\mathbb{R}\setminus\{0\}$.

We know sums, products, compositions, derivatives and antiderivatives of analytic functions are also analytic on the same domains, and in fact the Taylor series can be found as if the series were polynomials.

Taylor series

Write down the Taylor series centred at 0 for the following functions:

1. $\frac{x^2}{1+x}$
2. $(e^x)^2$
3. $\cos^2(x)$
4. $\ln(\frac{1+x}{1-x})$
5. $\frac{1}{(x^2-5x+6)}$

Evaluating series using Taylor series

We can prove at this point the following series converges. Taylor series also allow us to say what these converge to.

1. $\sum_{n=0}^{\infty} \frac{1}{2^n}$
2. $\sum_{n=0}^{\infty} \frac{n}{2^n}$
3. $\sum_{n=0}^{\infty} \frac{n^2}{2^n}$

Hint: Think of these as plugging in a specific x-value into a Taylor series. Think about the Taylor series you know and manipulate them (take derivatives, integrals, multiply by x etc.) to get something which is useful.