Today's topics and news

- Topics: The big theorem, improper Integral, comparison theorems

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- Homework: Watch video 13.1 - 13.9

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Definition

Let $\{a_n\}$, $\{b_n\}$ be (positive) sequences.

$a_n \ll b_n$ iff $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = 0$.

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True or false: If $\forall n \in \mathbb{N}$, $a_n \ll b_n$ then $\forall n \in \mathbb{N}$, $a_n < b_n$.

True or false: If $\forall n \in \mathbb{N}$, $a_n < b_n$, then $\forall n \in \mathbb{N}$, $a_n \ll b_n$.

True or false: If $a_n \ll b_n$ and $b_n \ll c_n$ then $a_n \ll c_n$

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True or false: If $\forall n \in \mathbb{N}$, $a_n < b_n$, then $\forall n \in \mathbb{N}$, $a_n \ll b_n$. False.

True or false: If $a_n \ll b_n$ and $b_n \ll c_n$ then $a_n \ll c_n$. True.

Applications of the big theorem

The big theorem for sequences

$$\ln(n) \ll n^a \ll c^n \ll n! \ll n^n$$

for every $a > 0$, $c > 1$

Compute:

1. $\lim_{n\to\infty} \frac{n!+e^n}{3n!+2e^n}$
2. $\lim_{n\to\infty} \frac{2^n+(2n)^2}{2^{n+1}+n^2}$
3. $\lim_{n\to\infty} \frac{n!+e^n}{n^n+n!}$

Improper Integrals

Type 1 improper integrals

Given a continuous function f on $[a, \infty)$, we define the improper integral

$$\int_a^\infty f(x)dx = \lim_{c\to\infty} \int_a^c f(x)dx.$$

Type 2 improper integrals

Given a function which is continuous on $(a, b]$, but which is not bounded on $(a, b]$, we define the improper integral

$$\int_a^b f(x)dx = \lim_{c\to a^+} \int_c^b f(x)dx.$$

We say these improper integrals converge if these respective limits converge, otherwise we say the improper integrals diverge.

The most important improper integrals

What types of improper integrals are these? Check if these integrals converge or diverge. Note the answer will depend on p.

1. $\int_1^{\infty} x^p dx$
2. $\int_0^1 x^p dx$

Doubly improper integrals

There are integrals which are improper for multiple reasons. For example:

1. $\int_{-\infty}^{\infty} x dx$.
2. $\int_{0}^{\infty} \frac{1}{x^2-3x+2} dx$.

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The first one doesn't fall into a type 1 integral because type 1 integrals are for continuous function over $(-\infty, a]$ or $[a, \infty)$. It's tempting to define

$\int_{-\infty}^{\infty} x dx = \lim_{A\to\infty} \int_{-A}^{A} x dx$ but this is not the way we do it.

Instead, we split any integral of this type into integrals over regions that are purely type 1 or type 2, and then say the improper integral diverges if any of the improper integrals we've split into fails to converge.

True or false

Given f and g continuous on $[0, \infty)$, if $\int_0^\infty f(x)dx$ diverges then $\int_0^\infty (f(x) + g(x))dx$ also diverges.

Given f continuous on $(0, \infty)$, if $\int_0^1 f(x)dx$ diverges then $\int_0^\infty f(x)dx$ also diverges.

Given f continuous on $[0, \infty)$, if $\int_1^\infty f(x)dx$ converges then $\int_0^\infty f(x)dx$ also converges.

True or false

Let $a \in \mathbb{R}$.
Let f and g be continuous on $[a, \infty)$.
Assume $\forall x \geq a, f(x) \leq g(x)$.
Are the following statements true or false? Justify your answer.

1. If $\int_a^\infty f(x)dx$ converges, then $\int_a^\infty g(x)dx$ converges.
2. If $\int_a^\infty f(x)dx = \infty$, then $\int_a^\infty g(x)dx = \infty$.
3. If $\int_a^\infty g(x)dx$ converges, then $\int_a^\infty f(x)dx$ converges.
4. If $\int_a^\infty g(x)dx = \infty$, then $\int_a^\infty f(x)dx = \infty$.

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True or false

Let $a \in \mathbb{R}$.
Let f and g be continuous on $[a, \infty)$.
Assume $\exists M \geq a$ s.t. $\forall x \geq M, 0 \leq f(x) \leq g(x)$.
Are the following statements true or false? Justify your answer.

1. If $\int_a^\infty f(x)dx$ converges, then $\int_a^\infty g(x)dx$ converges.
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BCT

Write a version of BCT for negative functions.

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Write a version of BCT for negative functions.

BCT also works the same way for type 2 improper integrals. Here you care about the inequality when you are near the "problematic" point.

Converge or diverge?

1. $\int_1^\infty \frac{1}{x^2+1000}\,dx$.
2. $\int_2^\infty \frac{1}{x^2-2}\,dx$.
3. $\int_1^\infty \frac{1}{x+e^x}\,dx$.

Limit Comparison Test

Let $a \in \mathbb{R}$,

Let f and g be continuous, positive functions on $[a, \infty)$.

Assume $\lim\limits_{x \to \infty} \frac{f(x)}{g(x)}$ exists and equals ______________________________, then,

______________________________ iff ______________________________.

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Let f and g be continuous, positive functions on $[a, \infty)$.

Assume $\lim_{x \to \infty} \frac{f(x)}{g(x)}$ exists and equals ______________________, then,

______________________ iff ______________________.

There are similar statements for negative functions and for type 2 improper integrals.

LCT generalization

Let $a \in \mathbb{R}$,

Let f and g be continuous, positive functions on $[a, \infty)$.

Suppose $\lim_{x \to \infty} \frac{f(x)}{g(x)} = 0$, what can you conclude?

Prove your guess. Hint: use BCT.

Converge or diverge?

1. $\int_2^\infty \frac{1}{x^2-2}dx$
2. $\int_1^\infty \frac{4x^2+2x+1}{\sqrt{x^7+3x^2+4}}dx$
3. $\int_2^3 \frac{1}{x\sqrt{x^2-4}}dx$
4. $\int_0^1 \frac{\sin(x)}{x^{\frac{3}{2}}}dx$
5. $\int_0^1 \cot(x)dx$
6. $\int_2^\infty \frac{(\ln(x))^{99}}{x^2}dx$
7. $\int_0^\infty e^{-x^2}dx$

8 $\int_0^1 \frac{1}{x+\sqrt{x}}dx$