Today's topics and news

- Topics: Sequences, (B)MCT and the big theorem
- Homework: Watch video 12.1, 12.4, 12.7 - 12.10 (12.1 - 12.6 recommended)
- Term test 3 takes place Monday 6:00 - 8:00 PM. Check BB for more details. Please contact us if you need to arrange an early sitting (at 3:30 - 5:30).

Definition

Write down the definition of the following statements:

1. a_n converges to a.
2. a_n converges.
3. a_n diverges.
4. a_n diverges to infinity (we also say converges to infinity for some reason:/).

Properties

All the usual properties you know for limits of functions more or less applies to limits of sequences. In particular, (provided the RHS limits exist):

1. $\lim_{n\to\infty}(a_n + kb_n) = \lim_{n\to\infty} a_n + k \lim_{n\to\infty} b_n$.
2. $\lim_{n\to\infty}(a_n b_n) = \lim_{n\to\infty} a_n \lim_{n\to\infty} b_n$.
3. $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$ provided $\lim_{n\to\infty} b_n \neq 0$.
4. If f is continuous, then $\lim_{n\to\infty} f(a_n) = f(\lim_{n\to\infty} a_n)$.
5. If $a_n \leq b_n \leq c_n$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n$, then $\lim_{n\to\infty} b_n$ exists and equal the two limits.

The proofs of all of these are very similar to the corresponding proofs in the function case. Homework: Try proving a couple.

Warm-up

Compute:

1. $\lim_{n\to\infty} \frac{\sin(n)}{n}$.
2. $\lim_{n\to\infty} \frac{n^2+2n+1}{3n^2-1}$.
3. $\lim_{n\to\infty} \frac{n^2+2n+1}{3n^3-1}$.
4. $\lim_{n\to\infty} \frac{n^3+2n+1}{3n^2-1}$.
5. $\lim_{n\to\infty} n - \sqrt{n^2+4n}$.

Warm-up

1. If $\lim_{n\to\infty} a_n = 100$, what is $\lim_{n\to\infty} a_{n+3}$?
2. True or false: If a_n converges and b_n diverges, then $a_n + b_n$ diverges.
3. True or false: If a_n diverges and b_n diverges, then $a_n + b_n$ diverges.

Given a function f, $\forall n \in \mathbb{N}$, define $a_n = f(n)$.

4. True or fase: If we know $\lim_{n\to\infty} a_n = a$, then $\lim_{x\to\infty} f(x) = a$.
5. True or false: If we know $\lim_{x\to\infty} f(x) = a$, then $\lim_{n\to\infty} a_n = a$.

Warm-up

1. If $\lim_{n\to\infty} a_n = 100$, what is $\lim_{n\to\infty} a_{n+3}$? 100

2. True or false: If a_n converges and b_n diverges, then $a_n + b_n$ diverges. True.

3. True or false: If a_n diverges and b_n diverges, then $a_n + b_n$ diverges. False.

Given a function f, $\forall n \in \mathbb{N}$, define $a_n = f(n)$.

4. True or fase: If we know $\lim_{n\to\infty} a_n = a$, then $\lim_{x\to\infty} f(x) = a$. False.

5. True or false: If we know $\lim_{x\to\infty} f(x) = a$, then $\lim_{n\to\infty} a_n = a$. True.

Note if you have a sequence defined this way and f itself satisfies the conditions of L'Hopital's rule, then you can apply it to find the limit of f and then use the second statement to say the limit of a_n must be the same.

Proof from definition

Prove from definition that $\lim_{n\to\infty} \frac{\sqrt{n+1}}{\sqrt{n}} = ?$

Proof from definition

Prove from definition that if $\lim_{n \to \infty} a_n = a$ then $\lim_{n \to \infty} a_n^2 = a^2$.

You've already done very similar proofs for functions and this is not very different.

Proof from definition

Prove from definition that if $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$ then $\lim_{n\to\infty}(2a_n + b_n) = 2a + b$

MCT

Write the following definitions: We say a sequence $\{a_n\}$ is **bounded above** iff

We say a sequence $\{a_n\}$ is **increasing** iff

We say a sequence $\{a_n\}$ is **non-decreasing** iff

We say a sequence $\{a_n\}$ is **eventually non-decreasing** iff

MCT: A sequence $\{a_n\}$ which is bounded above and eventually non-decreasing converges.

I can put "eventually" in front of the "bounded above" too but does that matter?

There is a similar statement for bounded below and eventually non-increasing.

Application of MCT

We define the sequence a_n recursively as follows:

$a_0 = \sqrt{2}$

$a_{n+1} = \sqrt{2 + a_n} \ \forall n \in \mathbb{N}$

1. Guess whether a_n is increasing or decreasing. Don't try to prove it yet.
2. If a_n does converge to some a, taking limits of the recursive relation, what must a be? (Keep in mind this is completely hypothetical as you have not yet proved that a_n converges.)
3. Guess an upper bound and a lower bound for a_n using 1 and 2, which is needed for MCT to work?
4. Prove your guess in 1.
5. Prove your bound in 3.
6. Does a_n converge? If so what does it converge to?

bounded, montone, convergent

Give examples of sequences which satisfy the following or justify they don't exist.

1. decreasing, bounded below, convergent.
2. decreasing, bounded below, divergent.
3. decreasing, not bounded below, convergent.
4. decreasing, not bounded below, divergent.
5. not decreasing, bounded below, convergent.
6. not decreasing, bounded below, divergent.
7. not decreasing, not bounded below, convergent.
8. not decreasing, not bounded below, divergent.

True or false

Every convergent sequence is eventually monotone, that is, eventually increasing or decreasing.

Every convergent sequence is bounded.

If $\lim_{n\to\infty} a_n = L$ then $\lim_{n\to\infty} a_{n^3} = L$.

If $\lim_{n\to\infty} a_{2n} = L$ then $\lim_{ng\to\infty} a_n = L$.

a_n converges and every number in the sequence is an integer is an integer. What can you say about the sequence?.

True or false

Every convergent sequence have to be eventually monotone, that is, eventually increasing or decreasing. False.

Every convergent sequence have to be bounded. True.

If $\lim_{n\to\infty} a_n = L$ then $\lim_{n\to\infty} a_{n^3} = L$. True.

If $\lim_{n\to\infty} a_{2n} = L$ then $\lim_{n\to\infty} a_n = L$. False.

a_n converges and every number in the sequence is an integer. What can you say about the sequence?. It's eventually contant.

Much less than

We use the notation $a_n \ll b_n$ to say the sequence a_n is much smaller than b_n.

Definition

Let $\{a_n\}$, $\{b_n\}$ be (positive) sequences.

$a_n \ll b_n$ iff $\lim_{n \to \infty} \frac{a_n}{b_n} = 0$.

True or false: If $\forall n \in \mathbb{N}$, $a_n \ll b_n$ then $\forall n \in \mathbb{N}$, $a_n < b_n$.

True or false: If $\forall n \in \mathbb{N}$, $a_n < b_n$, then $\forall n \in \mathbb{N}$, $a_n \ll b_n$.

True or false: If $a_n \ll b_n$ and $b_n \ll c_n$ then $a_n \ll c_n$

Much less than

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True or false: If $\forall n \in \mathbb{N}$, $a_n \ll b_n$ then $\forall n \in \mathbb{N}$, $a_n < b_n$. False.

True or false: If $\forall n \in \mathbb{N}$, $a_n < b_n$, then $\forall n \in \mathbb{N}$, $a_n \ll b_n$. False.

True or false: If $a_n \ll b_n$ and $b_n \ll c_n$ then $a_n \ll c_n$. True.

Applications of the big theorem

Compute:

1. $\lim\limits_{n\to\infty} \frac{n!+e^n}{3n!+2e^n}$
2. $\lim\limits_{n\to\infty} \frac{2^n+(2n)^2}{2^{n+1}+n^2}$
3. $\lim\limits_{n\to\infty} \frac{n!+e^n}{n^n+n!}$