Today's topics and news

- Topics: Indefinite integral, FTC I and II, integration by substitution
- Homework: Watch videos 9.6, 9,10, 9.11, 9.13 (everything else from 9.5 - 9.14 is optional).

Three expressions involve $\int$

Given $a, b \in \mathbb{R}$, remind youreslves what the following are called, the type of objects these are, and how they are defined:

1. $\int_a^b f(x)dx$
2. $\int_a^x f(t)dt$
3. $\int f(x)dx$

Are these the same as

1. $\int_a^b f(t)dt$
2. $\int_a^t f(x)dx$ or $\int_a^x f(x)dx$
3. $\int f(t)dt$?

Antiderivative

Antiderivative

Given $f(x)$ on $[a, b]$, we say $F(x)$ is an **antiderivative** of $f(x)$ on $[a, b]$ iff $F(x)$ is continuous on $[a, b]$, differentiable on (a, b), and $\forall x \in (a, b)$

$$F'(x) = f(x).$$

This definition can be made sense for open-ended intervals as well. I will often omit the intervals.

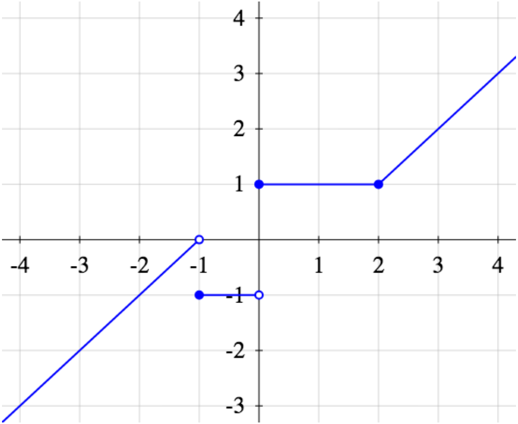
Antiderivatives

Find the antiderivatives of the following functions (by observation, guess and check):

1. $\int \frac{1}{3x+2} dx$
2. $\int \tan^2 \theta d\theta$
3. $\int \frac{1}{1+x^2} dx$
4. $\int \frac{1}{3+5t^2} dt$

Finding the accumulation function

Let $f(t)$ be the function expressed by the graph below. Define $F(x) = \int_1^x f(t)dt$, draw the graph of F.



FTC I

Fundamental Theorem of Calculus I

Given f is **continuous** on some interval $\mathbb{I}$ and $a \in \mathbb{I}$,

define $F(x) = \int_a^x f(t)dt \ \forall x \in \mathbb{I}$,

then F is **differentiable** and $F'(x) = f(x)$.

This lets us differentiate **accumulation functions**. It tells us that for a continuous function any function defined from its integrals is an antiderivative. This allows us to go from $\int_a^x$ to $\int$.

Counterexample

Give an example where FTC I fails if f is only assumed to be integrable.
Hint: What discontinuous, integrable functions do you know?

True or false

Let f and g be differentiable on $\mathbb{R}$. Assume that $\forall x \in \mathbb{R}, f'(x) = g(x)$. Which of the following statements are always true?

1. $f(x) = \int_0^x g(t)dt$
2. If $f(0) = 0$, then $f(x) = \int_0^x g(t)dt$
3. If $g(0) = 0$, then $f(x) = \int_0^x g(t)dt$
4. $\exists c \in \mathbb{R}$ s.t. $f(x) = c + \int_1^x g(t)dt$.

Using FTC I

Define

$$H_1(x) = \int_0^x \frac{1}{1+\sqrt{|t|}}dt,$$

$$H_2(x) = \int_0^{x^2} \frac{1}{1+\sqrt{|t|}}dt,$$

$$H_3(x) = \int_{2x}^{x^3-4} \frac{x}{1+\sqrt{|t|}}dt.$$

Find $H_1'(x), H_2'(x), H_3'(x)$.

Hint for the last two: FTC I only lets you differentiate accumulation functions and these are not accumulation functions. Write H_2 and H_3 using the accumulation function H_1 somehow and then differentiate.

Example

Suppose f is a continuous function and that

$$\int_0^x tf(t)dt = xsin(x) + cos(x) - 1.$$

Find $f(x)$.

FTC II

FTC II

Given f integrable over $[a, b]$,

suppose F is continuous on $[a, b]$ and F'(x) = f(x) on (a, b) (i.e. F is an antiderivative of f on $[a, b]$), then

$\int_a^b f(x)dx = F(b) - F(a) = F(x)|_{x=a}^{x=b}$

FTC II allows us to go from $\int$ to $\int_a^b$.

FTC II

FTC II (weaker version)

Given f continuous over $[a, b]$,

suppose F is continuous on $[a, b]$ and F'(x) = f(x) on (a, b) (i.e. F is an antiderivative of f on $[a, b]$), then

$\int_a^b f(x)dx = F(b) - F(a) = F(x)|_{x=a}^{x=b}$

The stronger version is harder to prove. The weaker version is basically a corollary of FTC I.

Using FTC II

FTC II turns the task of finding definite integrals into the task of finding antiderivatives and then evaluating on endpoints. The alternative would be to evaluate the limit of Riemann sums which is time-consuming.

From now on, we will almost always use FTC II in any definite integral problem we do. After this lecture, we will not always address the use of FTC II in computations.

Compute the following definite integrals using FTC II:

1. $\int_1^2 ((2x)^2 + 2x)dx$.
2. $\int_0^{\frac{\sqrt{3}}{2}} \frac{1}{\sqrt{1-t^2}} dt$
3. $\int_1^2 2^x(2 + 3^x)dx$
4. $\int_1^2 [\frac{d}{dx}(\frac{\sin(x)^2}{1+\arctan(x)^2+e^{-x^2}})]dx$

Area

Draw the graph of $f(x) = x$.

1. Using what you know about the area of triangles, find the area between $f(x)$ and the x-axis on $x \in [-2, 1]$.

2. Compute $\int_{-2}^{1} x dx$.

3. Why are these two numbers different?

4. Write the area between $f(x)$ and the x-axis on $x \in [-2, 1]$ using (several) definite integrals of $f(x) = x$.

5. Write the area between $f(x)$ and the x-axis on $x \in [-2, 1]$ using a single integral.

Area

What is the area between $f(x) = x^2$ and $g(x) = x$ on $x \in [-1, 1]$?

When I say what is the area between $f(x) = x^2$ and $g(x) = x$, without specifying x in some interval, what do I mean by this?

Integration by substitution

Compute:

1. $\int_2^3 \frac{1}{x\sqrt{\ln(x)}}dx$.
2. $\int_0^7 x^2(x+1)^{\frac{1}{3}}dx$.
3. $\int \cot(x)\ln(\sin(x))dx$.
4. $\int_0^1 \frac{x}{1+x^4}dx$.

Exercise

If f is integrable and $\int_0^4 f(x)dx = 4$, find $\int_0^2 xf(x^2)dx$.

Odd functions

Given an odd, integrable function $f(x)$ defined on $\mathbb{R}$, show $\forall a \in \mathbb{R}$,

$$\int_{-a}^{a} f(x)dx = 0.$$

Hint: Look at the integral as the sum of integral on two intervals and use substitution to show they cancel. (The fact that they should cancel should tell you what two intervals to choose.)