Today's topics and news

- Topics: Integrability, Riemann integrals and indefinite integrals
- Homework: Watch videos 8.3 - 8.7, 9.1 (9.2 - 9.4)

Joining Parititons

Let P and Q be two partitions of the $[a, b]$. Assume

1. $L_P(f) = 2$
2. $U_P(f) = 6$
3. $L_Q(f) = 3$
4. $U_Q(f) = 8$

1. Does $P \subseteq Q$?
2. Does $Q \subseteq Q$?
3. Bound $L_{P \cup Q}(f)$ and $U_{P \cup Q}(f)$.

ϵ-reformulation of integrability

Given a bounded function f on $[a, b]$,
f is integrable on $[a, b]$ iff $\forall\epsilon > 0$, $\exists P$ partition of $[a, b]$ s.t.
$U_P(f) - L_P(f) < \epsilon$.

Proof of $\rightarrow$:
Suppose f is integrable on $[a, b]$, given $\epsilon > 0$,
1. Is it true that are two possibly different partitions P_1 and P_2 satisfying $U_{P_1}(f) - L_{P_2}(f) < \epsilon$?
2. How can you come with a single partition which will satisfy $U_P(f) - L_P(f) < \epsilon$?

Proof of $\leftarrow$:
Suppose for $\epsilon > 0$, $\exists P$ partition of $[a, b]$ s.t. $U_P(f) - L_P(f) < \epsilon$.
1. Write down the relationship between $U_P(f), L_P(f), \overline{I_a^b}(f)$ and $\underline{I_a^b}(f)$
2. If $U_P(f) - L_P(f) < \epsilon$, what can you say about $\overline{I_a^b}(f) - \underline{I_a^b}(f)$?

Example: Warm-up

Consider

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{if } 0 < x \leq 1 \end{cases}$$

1. Let $P = \{0, 0.1, 0.5, 1\}$, what is $L_P(f)$? $U_P(f)$?
2. For an arbitrary partition $P = \{0 = x_0 < x_1 < ... < x_n = 1\}$, what is $L_P(f)$? $U_P(f)$?
3. Find a partition so that $L_P(f) = 0.99999$.
4. What is $\overline{I_0^1}(f)$? $\underline{I_0^1}(f)$?
5. Is f integrable on $[0, 1]$?

Example: Warm-up

Consider

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q} \\ 2 & \text{if } x \in \mathbb{R} \backslash \mathbb{Q} \end{cases}$$

Consider f on the interval [0,1],

1. Let $P = \{0, 0.1, 0.5, 1\}$, what is $L_P(f)$? $U_P(f)$?
2. For any $P = \{0 = x_0 < x_1 < ... < x_n = 1\}$, the lower sum $L(f, P)$ is equal to $L(h, P)$ for some simpler function h, what is that function?
3. What is $\underline{I_b^a(f)}$?

Riemann sums

A **Riemann partition** is a partition where the subintervals are all the same size. A Riemann partition for an interval $[a, b]$ is determined completely by n, the number of intervals it divides $[a, b]$ into. Let P_n be the Riemann partition of $[a, b]$ dividing $[a, b]$ into n subintervals, what is P_n?

Given a bounded function f on $[a, b]$ and a positive integer n, we define the $n - th$ right Riemann sum of f on [a,b] to be,

$R_n(f) = \sum_{k=1}^{n} f(x_k)\Delta x$, where
$x_k = a + k\frac{b-a}{n}$ and $\Delta x = \frac{b-a}{n}$.

The right Riemann sum is a type of Riemann sum, where you choose to use the y-value over the right endpoint of subintervals as the height of your rectangles. You can similarly define left Riemann sums ($L_n(f)$, don't get notations mixed up with $L_P(f)$, the lower sum), midpoint Riemann sums ($M_n(f)$) etc.

Integrability

It is often tedious to check integrability from the definition or $\epsilon-$ reformulation. With some work one can prove integrability for a large class of functions.

Integrability condition

If f is continuous on $[a, b]$, then it is integrable on $[a, b]$.

Note the implication does not go the other way. (See first example)

Norm of a partition

Given a partition $P = \{a = x_0 < x_1 < ... < x_n = b\}$, The norm of the partition $||P|| = \max\{x_i - x_{i-1} : i = 1...n\}$.

Integral as a limit

If f is integrable on $[a, b]$ and we have a sequence of parittions $\{P_k\}_{k=1}^{\infty}$ s.t. $\lim\limits_{k\to\infty} P_k = 0$, then $\int_a^b f(x)dx = \lim\limits_{k\to\infty} U_{P_k}(f) = \lim\limits_{k\to\infty} L_{P_k}(f)$.

Computing integrals before FTC

Consider $f(x) = x^2$,

1. $f(x)$ is integrable on $[0, 1]$. Why? (Note: You can **prove** integrability using $\epsilon-$ reformulation, this is because $U_P(f) - L_P(f)$ has a particularly nice expression in this example for certain partitions.)
2. Write down the definition of $R_n(f)$ on $[0, 1]$, find a simple formula for it without the $\sum$ symbol using summation formulas.
3. What is $\int_0^1 x^2 dx$?

Integral Properties

See Video 7.12 for a list.

Given $\int_0^1 f(x)dx = 2,\quad \int_1^3 f(x)dx = 3,\quad \int_0^3 g(x)dx = 1.$

Is it true $f(x) \geq 0$ on $[0, 1]$?

Compute:

1. $\int_0^3 f(t)dt$
2. $\int_0^3 f(x)dt$
3. $\int_3^1 f(x)dx$
4. $\int_{-1}^{-3} f(x)dx$
5. $\int_0^3 f(x) - 3g(x)dx$

Interpreting limits of sums as integrals

Write the following limits as (definite) integrals:

1. $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{n+k}{n^2}$
2. $\lim_{n\to\infty} \sum_{k=1}^{n} \left(\left(1 + \frac{6k}{n}\right) + \left(2 + \frac{2k}{n}\right)^2 \right) \frac{4}{n}$

Indefinite integrals

Given $f(x)$, we denote the indefinite integral of f by $\int f(x)dx$, it is defined as the collection of antideratives of $f(x)$. Notice a priori this has nothing to do with the (definite) integrals we defined before. Compute (by guess and check):

1. $\int \frac{1}{3x+2}dx$
2. $\int \tan^2(\theta)d\theta$
3. $\int \frac{1}{1+x^2}dx$
4. $\int \frac{1}{3+5t^2}dt$