Today's topics and news

- Topics: Sigma, infimum and supremum, definition of the integral
- Homework: Watch videos 7.8 - 7.12, 8.1 - 8.2
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Sigma

Recall that $\sum$ is called sigma and is a notation used the denote sum.

Given $a_1, a_2, ...a_n \in \mathbb{R}$,
$\sum_{k=1}^{n} a_k = a_1 + a_2 + ... + a_n$.

Define $\forall k \in \mathbb{N}, a_k = 2k + 1$. Compute:

1. $\sum_{k=2}^{4} a_k$.
2. $\sum_{i=2}^{4} a_k$.
3. $\sum_{i=2}^{4} a_i$.

Sigma notation exercise

Write out what both sides of the equation are according to the definition of sigma. Check if these equalities are true.

Given $a_k,\ b_k \in \mathbb{R},\ c \in \mathbb{R}$.

1. $\sum_{k=1}^{n} 1 = 1$
2. $\sum_{k=1}^{n} (ca_k + b_k) = c \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$
3. $\sum_{k=1}^{n} (a_k b_k) = (\sum_{k=1}^{n} a_k)(\sum_{k=1}^{n} b_k)$

Sigma notaiton exercise

Consider the sum $15 + 21 + 27 + 33 + ...297 + 303$. Write this in $\sum$ notation.

We have the following formulas:

1. $\sum_{k=1}^{n} 1 = n$
2. $\sum_{k=1}^{n} k = \frac{(n)(n+1)}{2}$
3. $\sum_{k=1}^{n} k^2 = \frac{(n)(n+1)(2n+1)}{6}$

Compute the sum above using these formulas.

Double sums

Compute:

1. $\sum_{i=1}^{n}(\sum_{k=1}^{n} 1)$
2. $\sum_{i=1}^{n}(\sum_{k=1}^{i} 1)$
3. $\sum_{i=1}^{n}(\sum_{k=1}^{i} i)$
4. $\sum_{i=1}^{n}(\sum_{k=1}^{i} k)$
5. $\sum_{i=1}^{n}(\sum_{k=1}^{i} ik)$

Use the following formulas:

1. $\sum_{k=1}^{n} k = \frac{(n)(n+1)}{2}$
2. $\sum_{k=1}^{n} k^2 = \frac{(n)(n+1)(2n+1)}{6}$
3. $\sum_{k=1}^{n} k^3 = \frac{(n)^2(n+1)^2}{4}$

Infimum

Definition

Given a set $A \subseteq \mathbb{R}$ and $a \in \mathbb{R}$, we say:

1. a i a **lower bound** of A if ______________________________.
2. A is **bounded below** if ______________________________.
3. a is a **minimum** of A if ______________________________.
4. a is an **infimum** of A if ______________________________ and ______________________________.

Using these definitions, answer the following questions:

1. Does $\emptyset$ have a lower bound?
2. Does $\emptyset$ have a minimum?
3. Does $\emptyset$ have an infimum?

Warm up

Find the supremum, infimum, maximum and minimum of the following sets if they exist:

1. $A = [0, 1)$
2. $B = \{1, 2, 3\}$
3. $C = \left\{ \frac{1}{n} : n \in \mathbb{N}, n \neq 0 \right\}$

Prove $\inf(A) = 0$:
1. Check 0 a lower bound.
2. Suppose l is another lower bound of A, why does 0 have to be larger?

Prove $\sup(A) = 1$:
1. Check 1 an upper bound?
2. Suppose u is another upper bound of A, and assume it's less than 1, come up with a number in A (using u) which is for sure larger than u, therefore contradicting the fact that u is an upper bound.

Existence of sup and inf

Existence theorem for sup and inf

Given $A \subseteq \mathbb{R}$,
A has a supremum iff A is bounded above and $A \neq \emptyset$.
A has an infimum iff A is bounded below and $A \neq \emptyset$.

Equivalent definitions of supremum

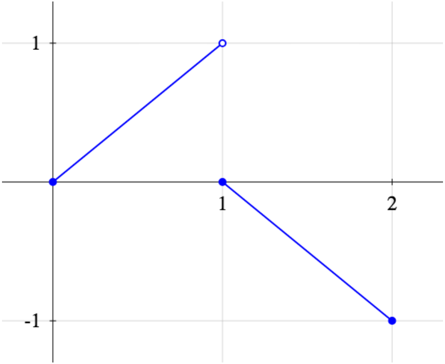
Assume u is an upper bound of the set A, which of the following statements are equivalent to $u = \sup(A)$?

1. $\forall v \leq u$, v is not an upper bound of A.
2. $\forall v < u$, v is not an upper bound of A.
3. $\forall v < u$, $\exists x \in A$ s.t. $v < x$.
4. $\forall v < u$, $\exists x \in A$ s.t. $v \leq x$.
5. $\forall v < u$, $\exists x \in A$ s.t. $v < x \leq u$.
6. $\forall v < u$, $\exists x \in A$ s.t. $v < x < u$.
7. $\forall \epsilon > 0$, $\exists x \in A$ s.t. $u - \epsilon < x \leq u$.
8. $\forall \epsilon > 0$, $\exists x \in A$ s.t. $u - \epsilon < x < u$.

Equivalent definition of sup

$u = \sup(A)$ iff u is an upper bound of A and $\forall \epsilon > 0$, $\exists x \in A$ s.t. $u - \epsilon < x \leq u$.

Sup and inf of functions



Compute:

1. $\sup_{x\in[0.5,1.5]} f(x)$
2. $\max_{x\in[0.5,1.5]} f(x)$
3. $\inf_{x\in[0.5,1.5]} f(x)$
4. $\min_{x\in[0.5,1.5]} f(x)$

Partitions

Which of the following are partitions of $[0, 2]$?

1. $[0, 2]$
2. $(0, 2)$
3. $\{0, 2\}$
4. $\{1, 2\}$
5. $\{0, 1, 1.5, 2\}$

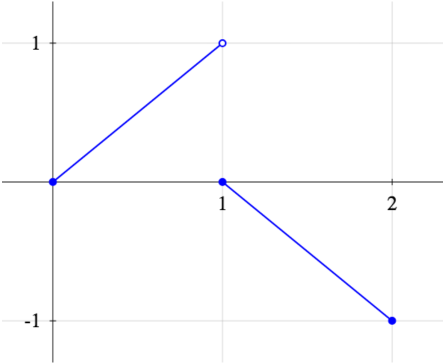
A **partition** of $[a, b]$ is **expressed** as a finite set S where $S \subseteq [a, b]$ and a, $b \in S$. It should be **thought of** as a way of dividing up the interval $[a, b]$, where you divide $[a, b]$ at all elements of S. Partitions are often written in order.

Definition of upper sum

Given a bounded **function** f on $[a, b]$ **and** a **partition** P, there are two ways to estimate the "signed area under the curve of f". These are called upper sum $U_P(f)$ and the lower sum $L_P(f)$. As you will see in the next slide, these estimates **depend** on the partition.

Given a partition $\{a = x_0 < x_1 < ...x_n = b\}$ of $[a, b]$ and a function f. Define $U_P(f)$.

Computing $U_P(f)$



Compute $U_P(f)$ for the following partitions:

1. $\{0, 2\}$
2. $\{0, 1, 2\}$
3. $\{0, 0.5, 1.5, 2\}$

Upper and lower integrals

We see that given a bounded function f on $[a, b]$ and a partition P of $[a, b]$, we can produce two estimates for the area under f between a and b, one of which is an underestimate and one of which is an overestimate.

There are infinitely many partitions, each giving their own (potentially) distinct over- and underestimates. If we want to get a true notion of the area, we can look all possible partitions and their corresponding underestimates $L_P(f)$, and think about the "largest" of all of them. We can similarly look at the all possible partitions and their corresponding overestimates $U_P(f)$, and think about the "smallest" of all of them. This motivates the following definitions:

1. The upper integral $\overline{I_a^b}(f) :=$
2. The lower integral $\underline{I_a^b}(f) :=$

Integrability

1. The upper integral $\overline{I_a^b}(f) := \inf(\{U_P(f) : \text{P is a partition of } [a,b]\})$
2. The lower integral $\underline{I_a^b}(f) := \sup(\{L_P(f) : \text{P is a partition of [a,b]}\})$

Note there is a relationship between these two numbers: $\overline{I_a^b}(f) \geq \underline{I_a^b}(f)$.

These are the best possible candidates for area under f and there's no preference for one over the other. That's why if they are not equal (i.e. $\overline{I_a^b}(f) > \underline{I_a^b}(f)$), we don't have a good notion of area and we say the function is not integrable. And if they are equal, then the function is **integrable** and

$$\int_a^b f(x)dx = \overline{I_a^b}(f) = \underline{I_a^b}(f).$$

Why do we use inf instead of min in the definition of $\overline{I_a^b}(f)$?

Consider $f(x) = x$, it is true that $\overline{I_0^1}(f) = \frac{1}{2}$.

Is there a partition P such that $U_P(f) = \overline{I_0^1}(f)$?

Lower and upper sums

Let f be a increasing, bounded function on $[a, b]$, and $P = \{a = x_0 < x_1 < ... < x_N = b\}$ a partition of $[a, b]$.

What is M_i and m_i in this case? Write down the formula for $U_P(f)$ and $L_P(f)$.