Announcements

- **Topic:** More about continuity, composition
- **Homework:** Watch videos 2.19 and 2.20.

Which one is the correct claim?

Claim 1?

(Assuming these limits exist)

$$\lim_{x \to a} g(f(x)) \;=\; g\left(\lim_{x \to a} f(x)\right)$$

Claim 2?

IF (A) $\lim_{x \to a} f(x) = L$, and (B) $\lim_{t \to L} g(t) = M$

THEN (C) $\lim_{x \to a} g(f(x)) = M$

A difficult example

Construct a pair of functions f and g such that

$$\begin{aligned} \lim_{x \to 0} f(x) &= 1 \\ \lim_{t \to 1} g(t) &= 2 \\ \lim_{x \to 0} g(f(x)) &= 42 \end{aligned}$$

Behaviour of limits under composition

Let g be a function defined on $\mathbb{R}$ and $f(x) = 4x$.

Assume $|x - 4| < 1 \implies |g(x) - g(4)| < 2$.

For which values of a and x can you guarantee $|g(f(x)) - g(f(a))| < 2$?

Behaviour of limits under composition

From previous examples, in general, it is not true that

$$\lim_{x \to a} f(g(x)) = f(\lim_{x \to a} g(x)).$$

However, the statement does become true if f is continuous (more specifically, at $\lim_{x \to a} g(x)$).

Theorem: limit "commutes" with continuous functions

IF $\lim_{x \to a} g(x)$ exists and f is continuous at $\lim_{x \to a} g(x)$.
THEN $\lim_{x \to a} f(g(x)) = f(\lim_{x \to a} g(x))$.

Behaviour of limits under composition

Prove the theorem. Assume for simplicity that f and g are defined on $\mathbb{R}$.

Theorem: limit "commutes" with continuous functions

IF $\lim_{x \to a} g(x)$ exists and f is continuous at $\lim_{x \to a} g(x)$.
THEN $\lim_{x \to a} f(g(x)) = f(\lim_{x \to a} g(x))$.

1. For simplicity of writing let L be $\lim_{x \to a} g(x)$. Write down your two assumptions in $\epsilon - \delta$ form.
2. Write down what you are trying to prove in $\epsilon - \delta$ form.
3. Prove it. Hint: Think about what the two assumptions give you control over.

Behaviour of limits under composition

Fill in the blank and then prove the claim.

Claim

Let $a, L \in \mathbb{R}$.

Let f be a function defined on a punctured neighbourhood of a (i.e. on some open neighbourhood of a, except possibly at a).

If $\lim_{x \to a} f(x) = L$

Then $\lim_{x \to \frac{a}{5}} 2f(5x) =$ _____.

Notice this theorem is not a consequence of the previous theorem becaues f is not assumed to be continuous.