Today's topics and news

- Topic: FTC 1 and 2

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- **Homework:** Watch videos 9.1 and 9.4 for Wednesday (9.2, 9.3 are supplementary).

Functions defined by integrals

Which ones of these are valid ways to define functions?

1. $F(x) = \int_0^x \frac{t}{1+t^8}\,dt$
2. $F(x) = \int_0^x \frac{x}{1+x^8}\,dx$
3. $F(x) = \int_0^x \frac{x}{1+t^8}\,dt$
4. $F(x) = \int_0^{x^2} \frac{t}{1+t^8}\,dt$
5. $F(x) = \int_{\sin x}^{e^x} \frac{t}{1+t^8}\,dt$
6. $F(x) = \int_0^3 \frac{t}{1+x^2+t^8}\,dt$
7. $F(x) = x\int_{\sin x}^{e^x} \frac{t}{1+x^2+t^8}\,dt$
8. $F(x) = t\int_{\sin x}^{e^x} \frac{t}{1+x^2+t^8}\,dt$

FTC I

Fundamental Theorem of Calculus 1

Given f is **continuous** on some open interval $\mathbb{I}$ and $a \in \mathbb{I}$,

Define $F(x) = \int_a^x f(t)dt$ $\forall$x$\in \mathbb{I}$,

Then F is **differentiable** and $F'(x) = f(x)$.

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This lets us differentiate **accumulation functions**. It tells us that for any continuous function f, the accumulation function $\int_a^x f(t)dt$ is an antiderivative of $f(x)$.

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FTC 1 allows us to relate $\int_a^x$ to $\int$ (i.e. $\int_a^x f(t)dt \in \int f(x)dx$).

Counterexample

Give an example where FTC 1 fails if f is only assumed to be integrable. Hint: What discontinuous, integrable functions do you know?

True or False?

Let f and g be differentiable functions with domain $\mathbb{R}$.
Assume that $f'(x) = g(x)$ for all x.
Which of the following statements must be true?

1. $f(x) = \int_0^x g(t)dt$.
2. If $f(0) = 0$, then $f(x) = \int_0^x g(t)dt$.
3. If $g(0) = 0$, then $f(x) = \int_0^x g(t)dt$.
4. There exists $C \in \mathbb{R}$ such that $f(x) = C + \int_0^x g(t)dt$.
5. There exists $C \in \mathbb{R}$ such that $f(x) = C + \int_1^x g(t)dt$.

Examples of FTC-1

Compute the derivative of the following functions

1. $F_1(x) = \int_0^1 e^{-t^2} dt.$
2. $F_2(x) = \int_0^x e^{-\sin t} dt.$
3. $F_3(x) = \int_1^{x^2} \frac{\sin t}{t^2} dt.$
4. $F_4(x) = \int_x^7 \sin^3\left(\sqrt{t}\right) dt.$
5. $F_5(x) = \int_{2x}^{x^2} \frac{1}{1+t^3} dt.$

A generalized version of FTC 1

Exercise

Let f be continuous and u, v be differentiable functions with domain $\mathbb{R}$. Let us call

$$F(x) = \int_{u(x)}^{v(x)} f(t)dt$$

Find a formula for

$$F'(x)$$

in terms of f, u, v, f', u', v'.

A generalized version of FTC 1 warning

WARNING: The previous slide outlines a **general method** to differentiate $\int_{u(x)}^{v(x)} f(t)dt$ and then gives **general result** to its derivative.

Do not use the **general result** directly on tests or assignments! We want to see that you are able to differentiate $\int_{u(x)}^{v(x)} f(t)dt$ using standard FTC 1 by thinking of it as compositions involving accumulation functions (i.e. actually be able to apply the **general method** of the previous slide.)

An integral equation

Assume f is a continuous function that satisfies, for every $x \in \mathbb{R}$:

$$\int_0^x e^t f(t)dt = \frac{\sin x}{x^2+1}$$

Find an explicit expression for $f(x)$.

FTC 2

FTC 2

Given f integrable over $[a, b]$,

Suppose F is continuous on $[a, b]$ and F'(x) = f(x) on (a, b) (i.e. F is an antiderivative of f on $[a, b]$), then $\int_a^b f(x)dx = F(b) - F(a) = F(x)|_a^b$

This turns the task of finding definite integrals into the task of finding antiderivatives and then evaluating on endpoints. The alternative would be to evaluate the limit of Riemann sums which is time-consuming.

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FTC 2 relates $\int$ to $\int_a^b$.

FTC 2

FTC 2 (weaker MAT137 version)

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Suppose F is continuous on $[a, b]$ and F'(x) = f(x) on (a, b) (i.e. F is an antiderivative of f on $[a, b]$), then

$\int_a^b f(x)dx = F(b) - F(a) = F(x)|_a^b$

The stronger version is harder to prove. The weaker version is basically a corollary of FTC I.

Definite integrals

Compute

1. $$\int_1^2 x^3 dx$$
2. $$\int_{1/2}^{1/\sqrt{2}} \frac{4}{\sqrt{1-x^2}} dx$$
3. $$\int_{\pi/4}^{\pi/3} \sec^2 x \, dx$$
4. $$\int_1^2 \left[\frac{d}{dx}\left(\frac{\sin^2 x}{1+\arctan^2 x + e^{-x^2}}\right)\right] dx$$

What is wrong?

$$\int_{-1}^{1} \frac{1}{x^4} dx \;=\; \left.\frac{-1}{3x^3}\right|_{-1}^{1} = \frac{-2}{3}$$

However, x^4 is always positive, so the integral should be positive.

Examples of FTC 2 — Areas

Calculate the area of the bounded region...

1. ... between the x-axis and $y = 4x - x^2$.
2. ... between $y = \cos x$, the x-axis, from $x = 0$ to $x = \pi$.
3. ... between $y = x^2 + 3$ and $y = 3x + 1$.
4. ... between $y = 1$, the y-axis, and $y = \ln(x + 1)$.