Today's topics and news

- Topic: Antiderivative, indefinite integrals
- **Homework:** Watch videos 8.3 - 8.7 for Tuesday and 9.1, 9.4 for Wednesday (9.2, 9.3 are supplementary).

The "ε–characterization" of integrability - First proof

Claim

Let f be a bounded function on $[a, b]$.

- IF f is integrable on $[a, b]$
- THEN "$\forall \varepsilon > 0$, $\exists$ a partition P of $[a, b]$, s.t. $U_P(f) - L_P(f) < \varepsilon$".

Proof:

Assume f is integrable on $[a, b]$ (i.e $\overline{I_a^b}(f) = \underline{I_a^b}(f)$). Let $\epsilon > 0$.

By ϵ-reformulation of sup and inf and the definition of $\overline{I_a^b}(f)$ and $\underline{I_a^b}(f)$, there exists partitions P_1 and P_2 s.t. $U_{P_1}(f) < \overline{I_a^b}(f) + \frac{\epsilon}{2}$ and $L_{P_2}(f) > \underline{I_a^b}(f) - \frac{\epsilon}{2}$.

Since $\overline{I_a^b}(f) = \underline{I_a^b}(f)$, $U_{P_1}(f) - L_{P_2}(f) < \epsilon$.

Choose $P = P_1 \cup P_2$. Then $U_{P_1}(f) \geq U_P(f) \geq L_P(f) \geq L_{P_2}(f)$. So $U_P(f) - L_P(f) \leq U_{P_1}(f) - L_{P_2}(f) < \epsilon$ as required.

The "ε–characterization" of integrability - Second proof

You are going to prove

Claim

Let f be a bounded function on $[a, b]$.

- IF "$\forall \varepsilon > 0$, $\exists$ a partition P of $[a, b]$, s.t. $U_P(f) - L_P(f) < \varepsilon$".
- THEN f is integrable on $[a, b]$

1. Recall the definition of "f is integrable on $[a, b]$".
2. For any partition P, order the quantities $U_P(f)$, $L_P(f)$, $\overline{I_a^b}(f)$, $\underline{I_a^b}(f)$.
3. For any partition P, order the quantities $U_P(f) - L_P(f)$, $\overline{I_a^b}(f) - \underline{I_a^b}(f)$, and 0.
4. Prove the following lemma "Let $a \geq 0$. IF $\forall \epsilon > 0$, $a < \epsilon$ THEN $a = 0$".
5. Use 3 and 4 together to conclude.
6. Write down a proof for the claim.

The "ε–characterization" of integrability

The "ε–characterization" of integrability

Let f be a bounded function on $[a, b]$.

f is integrable on $[a, b]$

IFF

$\forall \varepsilon > 0$, $\exists$ a partition P of $[a, b]$, s.t. $U_P(f) - L_P(f) < \varepsilon$.

Properties of the integral

Assume we know the following

$$\int_0^2 f(x)dx = 3, \quad \int_0^4 f(x)dx = 9, \quad \int_0^4 g(x)dx = 2.$$

Compute:

1. $\displaystyle\int_0^2 f(t)dt$
2. $\displaystyle\int_0^2 f(t)dx$
3. $\displaystyle\int_2^0 f(x)dx$
4. $\displaystyle\int_2^4 f(x)dx$
5. $\displaystyle\int_{-2}^0 f(x)dx$
6. $\displaystyle\int_0^4 [f(x) - 2g(x)]\ dx$

Riemann sums example

Imitate the calculation in Video 7.11.

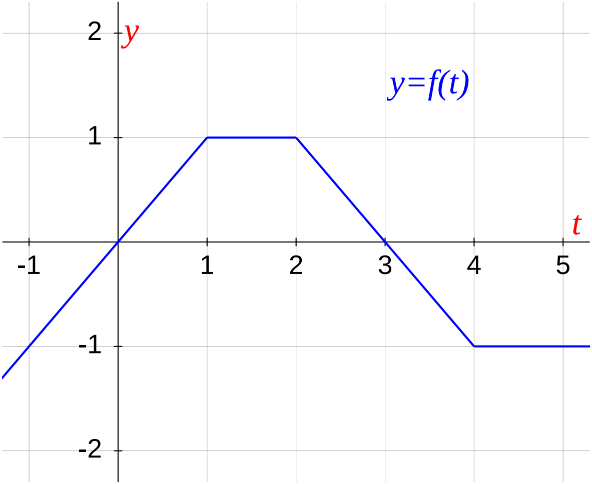
Exercise

Let $f(x) = x^2$ on $[0, 1]$.
Let $P_n = \{\text{breaking the interval into } n \text{ equal pieces}\}$.

1. Write a explicit formula for P_n.
2. What is Δx_i?
3. Write $S^*_{P_n}(f)$ as a sum when we choose x^*_i as the right end-point.
4. Add the sum
5. Compute $\lim_{n\to\infty} S^*_{P_n}(f)$.
6. Repeat the last 3 questions when we choose x^*_i as the left end-point.

Helpful formulas: $$\sum_{i=1}^{N} i = \frac{N(N+1)}{2}, \qquad \sum_{i=1}^{N} i^2 = \frac{N(N+1)(2N+1)}{6}$$

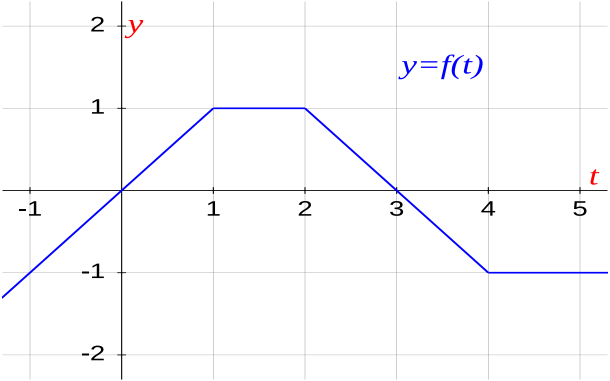
Towards FTC



Compute:

1. $\displaystyle\int_0^1 f(t)dt$
2. $\displaystyle\int_0^2 f(t)dt$
3. $\displaystyle\int_0^3 f(t)dt$
4. $\displaystyle\int_0^4 f(t)dt$
5. $\displaystyle\int_0^5 f(t)dt$

Towards FTC (continued)



Call $F(x) = \int_0^x f(t)dt$. This is a new function.

- Sketch the graph of $y = F(x)$.
- Using the graph you just sketched, sketch the graph of $y = F'(x)$.

Warm-up

Fix a, $b \in \mathbb{R}$. Given a function f integrable on $\mathbb{R}$.

Recall what type of object the following are and how they are defined.

1. $\int_a^b f(x)dx$
2. $\int_a^x f(t)dt$
3. $\int f(x)dx$

Which of the three are related from the definition?

Functions defined by integrals

Which ones of these are valid ways to define functions?

1. $F(x) = \int_0^x \frac{t}{1+t^8}\, dt$
2. $F(x) = \int_0^x \frac{x}{1+x^8}\, dx$
3. $F(x) = \int_0^x \frac{x}{1+t^8}\, dt$
4. $F(x) = \int_0^{x^2} \frac{t}{1+t^8}\, dt$
5. $F(x) = \int_{\sin x}^{e^x} \frac{t}{1+t^8}\, dt$
6. $F(x) = \int_0^3 \frac{t}{1+x^2+t^8}\, dt$
7. $F(x) = x\int_{\sin x}^{e^x} \frac{t}{1+x^2+t^8}\, dt$
8. $F(x) = t\int_{\sin x}^{e^x} \frac{t}{1+x^2+t^8}\, dt$

Antiderivatives

Compute these antiderivatives by guess-and-check.

1. $\int x^5 dx$
2. $\int \left(3x^8 - 18x^5 + x + 1\right) dx$
3. $\int \sqrt[3]{x}\, dx$
4. $\int \frac{1}{x^9}\, dx$
5. $\int \sqrt{x}\left(x^2 + 5\right) dx$
6. $\int \frac{1}{e^{2x}}\, dx$
7. $\int \sin(3x)\, dx$
8. $\int \cos(3x + 2)\, dx$
9. $\int \sec^2 x\, dx$
10. $\int \sec x \tan x\, dx$
11. $\int \frac{1}{x}\, dx$
12. $\int \frac{1}{x + 3}\, dx$