Today's topics and news

- Topic: Integrability, Riemann integrals
- **Homework:** Watch videos 8.1 and 8.2 for Wednesday.

Joining partitions

Assume

$$L_P(f) = 2, \quad U_P(f) = 6$$
$$L_Q(f) = 3, \quad U_Q(f) = 8$$

1. Is $P \subseteq Q$?
2. Is $Q \subseteq P$?
3. What can you say about $L_{P\cup Q}(f)$ and $U_{P\cup Q}(f)$?

Example 1: a constant function

Consider the function $f(x) = 2$ on $[0, 4]$.

1. Given $P = \{0, 1, e, \pi, 4\}$, compute $L_P(f)$ and $U_P(f)$.
2. Explicitly compute *all* the upper sums and *all* the lower sums.
3. Compute $\underline{I_0^4}(f)$
4. Compute $\overline{I_0^4}(f)$
5. Is f integrable on $[0, 4]$?

Example 2: a non-continuous function

Consider the function $f(x) = \begin{cases} 0 & x = 0 \\ 5 & 0 < x \leq 1 \end{cases}$, defined on $[0, 1]$.

1. Let $P = \{0, 0.2, 0.5, 0.9, 1\}$.
   Calculate $L_P(f)$ and $U_P(f)$ for this partition.
2. Fix an arbitrary partition $P = \{x_0, x_1, \ldots, x_N\}$ of $[0, 1]$.
   What is $U_P(f)$? What is $L_P(f)$? (Draw a picture!)
3. Find a partition P such that $L_P(f) = 4.99$.
4. What is the upper integral, $\overline{I_0^1}(f)$? (Don't worry about rigorously proving this for now)
5. What is the lower integral, $\underline{I_0^1}(f)$? (Don't worry about rigorously proving this for now)
6. Is f integrable on $[0, 1]$?

Example 3: a very non-continuous function

Consider the function $f(x) = \begin{cases} x^2 & x \in \mathbb{Q} \\ -1 & x \in \mathbb{Q}^c \end{cases}$, defined on $[0, 1]$.

1. Fix an arbitrary partition $P = \{x_0, x_1, \dots, x_N\}$ of $[0, 1]$.
   What is $L_P(f)$? What is $\underline{I_0^1}(f)$?
2. Come up with a (simpler) function g so that for any partition $P = \{x_0, x_1, \dots, x_N\}$ of $[0, 1]$,

$$U_P(g) = U_P(f)$$

   .
3. Using 2, make a guess as to what $\overline{I_0^1}(g) = \overline{I_0^1}(f)$ is. We can prove this using the theory of Riemann integrals.
4. Is f integrable on $[0, 1]$?

Warm-up

Let $A \subseteq \mathbb{R}$ and $a \in \mathbb{R}$. Give the ϵ-reformulation (or ϵ-characterization) definition of $\sup(A) = a$.

Note: This is not the original definition but one of the equivalent definitions I mentioned in Wednesday's lecture.

Are upper and lower integrals always realized by some partition?

Let f be an integrable function on $[a, b]$. Is it always true that there exist partitions P and Q of $[a, b]$ such that

$$L_P(f) = \underline{I_a^b}(f) = \overline{I_a^b}(f) = U_Q(f)?$$

The "ε–characterization" of integrability

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Let f be a bounded function on $[a, b]$.

f is integrable on $[a, b]$

IFF

$\forall \varepsilon > 0$, $\exists$ a partition P of $[a, b]$, s.t. $U_P(f) - L_P(f) < \varepsilon$.

The "ε–characterization" of integrability - First proof

You are going to prove

Claim

Let f be a bounded function on $[a, b]$.

- IF f is integrable on $[a, b]$
- THEN "$\forall \varepsilon > 0$, $\exists$ a partition P of $[a, b]$, s.t. $U_P(f) - L_P(f) < \varepsilon$".

1. Recall the definition of "f is integrable on $[a, b]$".
2. Write down the structure of the proof.
3. Fix $\varepsilon > 0$. Show there is a partition P s.t. $U_P(f) < \overline{I_a^b}(f) + \frac{\varepsilon}{2}$. Why?
4. Assume f is integrable on $[a, b]$. Fix $\varepsilon > 0$.
   Show there are partitions P_1 and P_2 s.t. $U_{P_1}(f) - L_{P_2}(f) < \varepsilon$.
5. Using P_1 and P_2 from the previous step, construct a partition P such that $U_P(f) - L_P(f) < \varepsilon$.
6. Write down a proof for the claim.

The "ε–characterization" of integrability - First proof

We are going to prove

Claim

Let f be a bounded function on $[a, b]$.

- IF f is integrable on $[a, b]$
- THEN "$\forall \varepsilon > 0$, $\exists$ a partition P of $[a, b]$, s.t. $U_P(f) - L_P(f) < \varepsilon$".

The "ε–characterization" of integrability - Second proof

You are going to prove

Claim

Let f be a bounded function on $[a, b]$.

- IF "$\forall \varepsilon > 0$, $\exists$ a partition P of $[a, b]$, s.t. $U_P(f) - L_P(f) < \varepsilon$".
- THEN f is integrable on $[a, b]$

1. Recall the definition of "f is integrable on $[a, b]$".
2. For any partition P, order the quantities $U_P(f)$, $L_P(f)$, $\overline{I_a^b}(f)$, $\underline{I_a^b}(f)$.
3. For any partition P, order the quantities $U_P(f) - L_P(f)$, $\overline{I_a^b}(f) - \underline{I_a^b}(f)$, and 0.
4. Write down a proof for the claim.

The "ε–characterization" of integrability

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Let f be a bounded function on $[a, b]$.

f is integrable on $[a, b]$

IFF

$\forall \varepsilon > 0$, $\exists$ a partition P of $[a, b]$, s.t. $U_P(f) - L_P(f) < \varepsilon$.

Properties of the integral

Assume we know the following

$$\int_0^2 f(x)dx = 3, \quad \int_0^4 f(x)dx = 9, \quad \int_0^4 g(x)dx = 2.$$

Compute:

1. $\int_0^2 f(t)dt$
2. $\int_0^2 f(t)dx$
3. $\int_2^0 f(x)dx$
4. $\int_2^4 f(x)dx$
5. $\int_{-2}^0 f(x)dx$
6. $\int_0^4 [f(x) - 2g(x)]\ dx$

Riemann sums example

Imitate the calculation in Video 7.11.

Exercise

Let $f(x) = x^2$ on $[0, 1]$.
Let $P_n = \{\text{breaking the interval into } n \text{ equal pieces}\}$.

1. Write a explicit formula for P_n.
2. What is Δx_i?
3. Write $S^*_{P_n}(f)$ as a sum when we choose x_i^* as the right end-point.
4. Add the sum
5. Compute $\lim_{n\to\infty} S^*_{P_n}(f)$.
6. Repeat the last 3 questions when we choose x_i^* as the left end-point.

Helpful formulas: $$\sum_{i=1}^{N} i = \frac{N(N+1)}{2}, \qquad \sum_{i=1}^{N} i^2 = \frac{N(N+1)(2N+1)}{6}$$

Riemann sums backwards

Interpret the following limits as integrals:

1. $$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n} \sin \frac{i}{n}$$

2. $$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{n+i}{n^2}$$

Fix these FALSE statements

1. Let f and g be bounded functions on $[a, b]$. Then

$$\begin{matrix}\text{sup of } (f+g) \\ \text{on } [a,b]\end{matrix} \quad = \quad \begin{matrix}\text{sup of } f \\ \text{on } [a,b]\end{matrix} \quad + \quad \begin{matrix}\text{sup of } g \\ \text{on } [a,b]\end{matrix}$$

2. Let $a < b < c$. Let f be a bounded function on $[a, c]$. Then

$$\begin{matrix}\text{sup of } f \\ \text{on } [a,c]\end{matrix} \quad = \quad \begin{matrix}\text{sup of } f \\ \text{on } [a,b]\end{matrix} \quad + \quad \begin{matrix}\text{sup of } f \\ \text{on } [b,c]\end{matrix}$$

3. Let f be a bounded function on $[a, b]$. Let $c \in \mathbb{R}$. Then:

$$\begin{matrix}\text{sup of } (cf) \\ \text{on } [a,b]\end{matrix} \quad = \quad c \left(\begin{matrix}\text{sup of } f \\ \text{on } [a,b]\end{matrix} \right)$$