Today's topics and news

- Topic: Definition of integrals
- **Homework:** Watch videos 7.8 - 7.12 for next Tuesday and watch videos 8.1, and 8.2 for Wednesday.

Equivalent definitions of supremum

Assume u is an upper bound of the set A, which of the following statements are equivalent to $u = \sup(A)$?

1. $\forall v \leq u$, v is not an upper bound of A.
2. $\forall v < u$, v is not an upper bound of A.
3. $\forall v < u$, $\exists x \in A$ s.t. $v < x$.
4. $\forall v < u$, $\exists x \in A$ s.t. $v \leq x$.
5. $\forall v < u$, $\exists x \in A$ s.t. $v < x \leq u$.
6. $\forall v < u$, $\exists x \in A$ s.t. $v < x < u$.
7. $\forall \epsilon > 0$, $\exists x \in A$ s.t. $u - \epsilon < x \leq u$.
8. $\forall \epsilon > 0$, $\exists x \in A$ s.t. $u - \epsilon < x < u$.

Partitions

Which of the following are partitions of $[0, 2]$?

1. $[0, 2]$
2. $(0, 2)$
3. $\{0, 2\}$
4. $\{1, 2\}$
5. $\{0, 1, 1.5, 2\}$

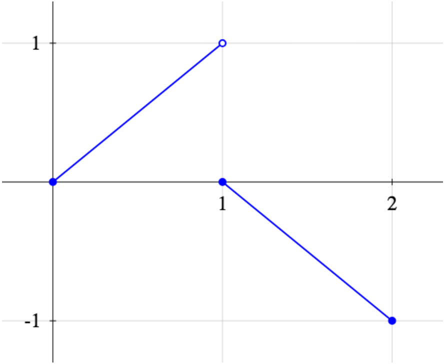
A **partition** of $[a, b]$ is **expressed** as a finite set S where $S \subseteq [a, b]$ and $a,\ b \in S$. It should be **thought of** as a way of dividing up the interval $[a, b]$ into finitely many subintervals, where you divide $[a, b]$ at all elements of S. Partitions are often written in order.

Definition of upper sum

Given a **bounded function** f on $[a, b]$ **and** a **partition** P, there are two ways to estimate the"signed area under the curve of f". These are called upper sum $U_P(f)$ and the lower sum $L_P(f)$. As you will see in the next slide, these estimates **depend** on the partition.

Given a partition $\{a = x_0 < x_1 < ...x_n = b\}$ of $[a, b]$ and a function f. Define $U_P(f)$.

Computing $U_P(f)$



Compute $U_P(f)$ for the following partitions:

1. $\{0, 2\}$
2. $\{0, 0.5, 1.5, 2\}$

Upper and lower integrals

We see that given a bounded function f on $[a, b]$ and a partition P of $[a, b]$, we can produce two estimates for the area under f between a and b, one of which is an underestimate and one of which is an overestimate.

There are infinitely many partitions, each giving their own (potentially) distinct over- and underestimates. If we want to get a true notion of the area, we can look all possible partitions and their corresponding underestimates $L_P(f)$, and think about the "largest" of all of them. We can similarly look at the all possible partitions and their corresponding overestimates $U_P(f)$, and think about the "smallest" of all of them. This motivates the following definitions:

1. The upper integral $\overline{I_a^b}(f) :=$
2. The lower integral $\underline{I_a^b}(f) :=$

Integrability

1. The upper integral $\overline{I_a^b}(f) := \inf(\{U_P(f) : \text{P is a partition of } [a, b]\})$
2. The lower integral $\underline{I_a^b}(f) := \sup(\{L_P(f) : \text{P is a partition of [a,b]}\})$

Note there is a relationship between these two numbers: $\overline{I_a^b}(f) \geq \underline{I_a^b}(f)$.

These are the best possible candidates for area under f and there's no preference for one over the other. That's why if they are not equal (i.e. $\overline{I_a^b}(f) > \underline{I_a^b}(f)$), we don't have a good notion of area and we say the function is not integrable. And if they are equal, then the function is **integrable** and

$$\int_a^b f(x)dx = \overline{I_a^b}(f) = \underline{I_a^b}(f).$$