Today's topics and news

- Topic: Continuity
- **Homework:** Watch videos 2.19 and 2.20 for next Wednesday.
- **Test 1** takes place next Friday from 4 - 6PM. If you have conflict please follow the the instructions on the test 1 instruction page on the main course website.

Another squeeze theorem

Let $a \in \mathbb{R}$. Let f and g be functions defined near a, except possibly at a.

IF

- $\exists p > 0$ s.t. $0 < |x - a| < p \implies f(x) \geq g(x)$.
- $\lim_{x \to a} g(x) = \infty$.

THEN

- $\lim_{x \to a} f(x) = \infty$.

Prove this theorem.

Hint: The proof of this theorem is similar to but easier than the standard squeeze theorem. Write down the relevant $M - \delta$ definitions and try to prove one from the other.

Continuity

For a function f defined on an open interval of a, we say $f(x)$ is cts at a iff

Definition 1

$\lim_{x \to a} f(x) = f(a)$

This is clearly equivalently to

Definition 2

$\forall \epsilon > 0, \exists \delta > 0$ s.t. $0 < |x - a| < \delta \implies |f(x) - f(a)| < \epsilon.$

Slightly less clearly, it is also equivalent to

Definition 3

$\forall \epsilon > 0, \exists \delta > 0$ s.t. $|x - a| < \delta \implies |f(x) - f(a)| < \epsilon.$

Continuity on different sets

Continuous at a point

f continuous at c means $\lim_{x \to c} f(x) = f(c)$.

Continuous on an open interval

f continuous on the interval (a, b) means $\forall c \in (a, b)$, f is continuous at c.

Continuous on a closed interval

f continuous on the interval $[a, b]$ means

1. $\lim\limits_{x \to a^+} f(x) = f(a)$
2. $\forall c \in (a, b)$, f is continuous at c
3. $\lim\limits_{x \to b^-} f(x) = f(b)$

Dirichlet function

Consider the Dirichlet function

$$D(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R} \backslash \mathbb{Q} \end{cases}$$

1. Write the definition of $\lim_{x \to 0} D(x) \neq 0.5$.
2. Prove it.
3. Write the definition of $\lim_{x \to 0} D(x)$ DNE.
4. Exercise: Prove 3.

Continuity examples

Find examples of a function defined on $\mathbb{R}$ satisfying the following conditions:

1. $f(x)$ is continuous on $\mathbb{R}$.
2. $g(x)$ is continuous at every $c \in \mathbb{R} \setminus \{0\}$ and discontinuous at 0.
3. $h(x)$ is discontinuous at every $c \in \mathbb{R}$.
4. $m(x)$ is continuous at 0 and discontinuous at every $c \in \mathbb{R}$.

Hint: Try adjusting the Dirichlet function.

Behaviour of limits under composition

From examples we've seen before, in general, it is not true that

$$\lim_{x \to a} f(g(x)) = f(\lim_{x \to a} g(x)).$$

However, the statement does become true if f is continuous (at $\lim_{x \to a} g(x)$).

Theorem: limit "commutes" with continuous functions

IF $\lim_{x \to a} g(x)$ exists and f is continuous at $\lim_{x \to a} g(x)$.
THEN $\lim_{x \to a} f(g(x)) = f(\lim_{x \to a} g(x))$.

Behaviour of limits under composition

Prove the theorem (assume for simplicity that f and g are defined on $\mathbb{R}$).

Theorem: limit "commutes" with continuous functions

IF $\lim_{x \to a} g(x)$ exists and f is continuous at $\lim_{x \to a} g(x)$.
THEN $\lim_{x \to a} f(g(x)) = f(\lim_{x \to a} g(x))$.

1. For simplicity of writing let L be $\lim_{x \to a} g(x)$. Write down your two assumptions in $\epsilon - \delta$ form.
2. Write down what you are trying to prove $\lim_{x \to a} f(g(x)) = f(L)$ in $\epsilon - \delta$ form.
3. Prove it. Hint: You are going to have to use the δ you get from one of your assumptions as the ϵ in the other assumption.