Today's topics and news

- Topic: Proofs from definition of a limit, limit laws
- **Homework:** Watch videos 2.12 and 2.13 for Wednesday.

Existence of limits

Given $a \in \mathbb{R}$.

Write down the definition of the following statments:

1. $\lim_{x \to a} f(x)$ exists.
2. $\lim_{x \to a} f(x)$ does not exist.

Given $a, L \in \mathbb{R}$.

Write down the definition of the following statments:

1. $\lim_{x \to a} f(x) = \infty$.
2. $\lim_{x \to \infty} f(x) = L$.

Hint: For 1, you want to replace the two parts with ϵ in the $\epsilon - \delta$ definition for limits. Instead of saying $f(x)$ gets arbitrarily close to L as x gets close to a, you want to say $f(x)$ gets arbitrarily large. How can you do this?

Warm-up

Suppose you know:

1. If $|x-2| < 4$, then S_1 is true.
2. If $|x-2| < 5$, then S_2 is true.

What condition do you need to guarantee S_1 and S_2 are both true?

Suppose you know:

1. If $x > 100$, then S_1 is true.
2. If $x > 1000$, then S_2 is true.

What condition do you need to guarantee S_1 and S_2 are both true?

Warm-up

1. Find a value of $\delta > 0$ s.t.

$$|x-2| < \delta \implies |2x-4| < 1.$$

2. Find all values of $\delta > 0$ s.t.

$$|x-2| < \delta \implies |2x-4| < 1.$$

3. Find all values of $\delta > 0$ s.t.

$$|x-2| < \delta \implies |2x-4| < 0.1.$$

3. Let $\epsilon > 0$, find all values of $\delta > 0$ s.t.

$$|x-2| < \delta \implies |2x-4| < \epsilon.$$

An $\epsilon - \delta$ proof

Goal

Prove

$$\lim_{x \to 2} 2x = 4$$

from the definition.

1. Write down the formal definition of claim. This is the statement you will need to prove.
2. Write down the structure of the proof without details.
3. Write down the complete proof.

Finding δ

1. Find a value of $\delta > 0$ s.t.

$$|x - 2| < \delta \implies |x + 2| < 10.$$

(What other δ will work?)

2. Let $\epsilon > 0$, find a value of $\delta > 0$ s.t.

$$|x - 2| < \delta \implies |x - 2| < \frac{\epsilon}{10}.$$

(What other δ will work?)

3. Let $\epsilon > 0$, find many values of $\delta > 0$ s.t.

$$|x - 2| < \delta \implies |x^2 - 4| < \epsilon$$

using your work from (1) and (2).

An $\epsilon - \delta$ proof

Goal

Prove

$$\lim_{x \to 2} x^2 = 4$$

from the definition.

1. Write down the formal definition of claim.
2. Write down the structure of the proof without details.
3. Write down the complete proof.

Steps for doing an $\epsilon - \delta$ proof rough work

Goal

Prove

$$\lim_{x \to 2} x^2 = 4$$

from the definition.

1. Write down the formal definition of claim.

2. Start with the $|f(x) - L|$ part of the defintion. Algebraically manipulate it to get several terms. $|x^2 - 4| = |x+2||x-2|$

3. Determine which one of the terms you can make arbitrarily small by constraining $|x - a|$. $|x - 2|$

4. Bound all other terms by (reasonable) constants by constraining $|x - a|$.The only other term is $|x + 2|$, we bounded this by 10 by constraining $|x - 2| < 6$.

Is this proof correct?

Claim

$\lim_{x \to 2} x^2 = 4$

Proof:

Let $\epsilon > 0$.
Choose $\delta = \frac{\epsilon}{|x+2|}$.
Let $x \in \mathbb{R}$.
Asssume $0 < |x - 2| < \delta$, then,

$$|x^2 - 4| = |x - 2||x + 2| < \frac{\epsilon}{|x + 2|}|x + 2| = \epsilon.$$

□

Homework: an $\epsilon - \delta$ proof

Goal

Prove

$$\lim_{x \to 4} \frac{3}{x} = \frac{3}{4}$$

from the definition.

1. Write down the formal definition of claim.

2. Start with the $|f(x) - L|$ part of the defintion. Algebraically manipulate it to get several terms.

3. Determine which one of the terms you can make arbitrarily small by constraining $|x - a|$.

4. Bound all other terms by (reasonable) constants by constraining $|x - a|$.

5. Write down a formal proof.

Limit laws warm-up

Are the following the same?

1. $\lim_{x \to 0^+} \frac{1}{x} - \lim_{x \to 0^+} \frac{1}{x}$
2. $\lim_{x \to 0^+} \left(\frac{1}{x} - \frac{1}{x}\right)$

Indeterminate form

Let $a \in \mathbb{R}$.
Let f and g be functions defined near a.
Assume $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$.

What can we conclude about $\lim_{x \to a} \frac{f(x)}{g(x)}$?

1. The limit is 1 the 0s cancel.
2. The limit does not exist because it's $\frac{0}{0}$.
3. We do not have enough information to decide.

True or false?

Claim

Let $a \in \mathbb{R}$.

Let f and g be functions defined near a.

- IF $\lim_{x \to a} f(x) = 0$,
- THEN $\lim_{x \to a} [f(x)g(x)] = 0$.