# Today's topics and news

- Topic: Analytic functions, New power series
- **Homework:** Watch videos 14.11 - 14.14 for Tuesday.

# Taylor Series

## Taylor Series: For $C^\infty$ functions

Let $a \in \mathbb{R}$. For $C^\infty$ functions $f(x)$ ,
$S(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$ is the Taylor series of $f$ centred $a$.

One of the reasons why we want to look at the Taylor series is in the hope that it would nicely approximate the function $f(x)$ near $a$. This is not always the case. We say

## Analyticity

A function $f$ is analytic at $a$ iff,
$S_a(x)$, the Taylor series expansion of $f$ around $a$ converges to $f(x)$ in a neighbourhood of $a$.
A function $f$ is analytic on some set $D$ iff $\forall a \in D$, it's analytic at $a$ .

Example: We have shown that $f(x) = \frac{1}{1-x}$ is analytic at 0.

# A pathological example

Consider the function $f(x)$ defined as follows:

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ e^{\frac{-1}{x^2}} & \text{if } x \neq 0 \end{cases}$$

One can show that this function is $C^\infty$ and $f^{(k)}(0) = 0$ $\forall k \in \mathbb{N}$.

1. Does this function have a Taylor polynomial of order 9 around 0?
2. Does this function have a Taylor Series around 0?
3. Is this function analytic?

# True or false

Let $f$ be a $C^\infty$ function defined on $\mathbb{R}$.

1. If $f$ is analytic at 0, then $\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = f(x)$.
2. If $f$ is analytic on $\mathbb{R}$, then $\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = f(x)$.

# Taylor series centred at $x = 1$

Write the Taylor series for $\frac{1}{1-x}$ centred around 0.5.

What is the interval of convergence of this Taylor series? Does it converge to $\frac{1}{1-x}$ on that interval?

Is $\frac{1}{1-x}$ analytic at 0.5?

**Lagrange's Remainder Theorem**

Suppose $f$ is $C^{n+1}$ on some interval $\mathbb{I}$ containing $a$.

Let $P_n$ be the $n^{th}$ Taylor Polynomial of $f$ at $a$.

Consider $R_n(x) = f(x) - P_n(x)$ the remainder,

then for any $x \in \mathbb{I}$, there exists $\xi$ between $a$ and $x$ s.t.

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}$$

Notice that $\xi$ depends on $n$ and $x$.

# Proving a function is analytic

We will now show that $\sin(x)$ is analytic on $\mathbb{R}$. To show a function $f$ is analytic on some $D$ we need to show that $f$ is analytic on each point $a \in \mathbb{D}$. This in turn means that the Taylor series of $f$ centred at $a$ converges to $f$ in a small neighbourhood of $a$.

We will call $S_a(x)$ the Taylor series for $\sin(x)$ centred around $a$. We will call $P_{n,a}(x)$ the $n^{th}$ Taylor polynomial for $\sin(x)$ centred around $a$.

1. We will show that for any $a, x \in \mathbb{R}$, $S_a(x)$ converges to $\sin(x)$. In other words, the small neighbourhood around each $a$ for which $S_a$ converges to $f$ can in fact be taken to be all of $\mathbb{R}$. Use Lagrange's Theorem to write down an expression for the remainder $R_{n,a}(x) = f(x) - P_{n,a}(x)$
2. Show that $\lim_{n\to\infty} R_{n,a}(x) = 0$.

This of course means $\sin(x)$ is analytic on $\mathbb{R}$.

# Taylor series gymnastics

Write the following functions in a way where you can easily find their Maclaurin series using series you already know.

1. $f(x) = \dfrac{x^2}{1+x}$
2. $f(x) = (e^x)^2$
3. $f(x) = \sin\left(2x^3\right)$
4. $f(x) = \cos^2 x$
5. $f(x) = \ln \dfrac{1+x}{1-x}$
6. $f(x) = \dfrac{1}{(1+x^2)(1+x)}$

*Note:* You do not need to take any derivatives. You can reduce them all to other Maclaurin series you know.

# Arctan

1. Write the function

$$f(x) = \arctan x$$

as a power series centered at 0.
*Hint:* Compute the first derivative. Then stop to think.

2. What is $f^{(2019)}(0)$?