Today's topics and news

- Topic: Limit comparison for improper integrals
- **Homework:** Watch videos 13.1 - 13.7 for Tuesday and 13.8, 13.9 for Wednesday.

True or False - Part III

Let $a \in \mathbb{R}$.
Let f and g be continuous functions on $[a, \infty)$.
Assume that $\boxed{\exists M \geq a \text{ s.t. } \forall x \geq M, \quad 0 \geq f(x) \geq g(x)}$.
What can we conclude?

1. IF $\int_a^\infty f(x)dx$ is convergent, THEN $\int_a^\infty g(x)dx$ is convergent.
2. IF $\int_a^\infty f(x)dx = -\infty$, THEN $\int_a^\infty g(x)dx = -\infty$.
3. IF $\int_a^\infty g(x)dx$ is convergent, THEN $\int_a^\infty f(x)dx$ is convergent.
4. IF $\int_a^\infty g(x)dx = -\infty$, THEN $\int_a^\infty f(x)dx = -\infty$.

(3) is called "eventual BCT". Of course there's a version for non-negative functions as well.

What can you conclude?

Let $a \in \mathbb{R}$. Let f be a continuous, positive function on $[a, \infty)$.

In each of the following cases, what can you conclude about $\int_a^\infty f(x)dx$?

Is it convergent, divergent, or we do not know?

1. $\forall b \geq a, \ \exists M \in \mathbb{R}$ s.t. $\int_a^b f(x)dx \leq M.$
2. $\exists M \in \mathbb{R}$ s.t. $\forall b \geq a, \int_a^b f(x)dx \leq M.$
3. $\exists M > 0$ s.t. $\forall x \geq a, \ f(x) \leq M.$
4. $\exists M > 0$ s.t. $\forall x \geq a, \ f(x) \geq M.$

A variation on LCT

This is the theorem you have learned:

Theorem (Limit-Comparison Test)

Let $a \in \mathbb{R}$. Let f and g be positive, continuous functions on $[a, \infty)$.

- IF the limit $L = \lim_{x \to \infty} \frac{f(x)}{g(x)}$ exists and $L > 0$.
- THEN $\int_a^\infty f(x)dx$ and $\int_a^\infty g(x)dx$ are both convergent or both divergent.

What if we change the hypothesis to $L = 0$?

1. Write down the new version of this theorem (different conclusion).
2. Prove it using BCT.

Hint: If $\lim_{x \to \infty} \frac{f(x)}{g(x)} = 0$, what is larger? $f(x)$ or $g(x)$?

Homework: Comparison tests for type 2 improper integrals

- A type-1 improper integral is an integral of the form $\int_a^\infty f(x)dx$, where f is a continuous, bounded function on $[c, \infty)$
- A type-2 improper integral is an integral of the form $\int_a^b f(x)dx$, where f is a continuous function on $(a, b]$ possibly with a vertical asymptote at a.

In the videos, you learned BCT and LCT for type-1 improper integrals.

1. Write the statement of the (standard and eventual) BCT for type-2 improper integrals.
2. Write the statement of the LCT for type-2 improper integrals.
3. Construct an example that you can prove is convergent thanks to one of these theorems.
4. Construct an example that you can prove is divergent thanks to one of these theorems.

Convergent or divergent?

1. $$\int_1^\infty \frac{x^3+2x+7}{x^5+11x^4+1}\,dx$$
2. $$\int_1^\infty \frac{1}{\sqrt{x^2+x+1}}\,dx$$
3. $$\int_0^1 \frac{3\cos x}{x+\sqrt{x}}\,dx$$
4. $$\int_0^1 \cot x\,dx$$
5. $$\int_0^1 \frac{\sin x}{x^{3/2}}\,dx$$
6. $$\int_0^\infty e^{-x^2}\,dx$$
7. $$\int_2^\infty \frac{(\ln x)^{10}}{x^2}\,dx$$
8. $$\int_0^1 \frac{\arcsin x}{x^{3/2}}\,dx$$

Harder example

For which values of $a > 0$ is the integral

$$\int_0^{\infty} \frac{\arctan x}{x^a}\, dx$$

convergent?