Today's topics and news

- Topic: Improper integrals, basic comparison for improper integrals

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- **Homework:** Watch videos 12.9 and 12.10 for Wednesday.

Refining the Big Theorem

We know from the Big Theorem that

The Big Theorem

$$ln(n) \ll n^a \ll c^n \ll n! \ll n^n$$

for every $a > 0, c > 1$.

There are actually lots of sequences in between each of these.

Exercise: Construct new sequencse $\{u_n\}$, $\{v_n\}$, $\{w_n\}$, $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ such that, for every $a > 0$ and for every $c > 1$,

$$u_n \ll ln(n) \ll v_n \ll n^a \ll w_n \ll c^n \ll x_n \ll n! \ll y_n \ll n^n \ll z_n$$

Refining the Big Theorem

1. Construct a sequence $\{u_n\}_n$ such that

$$\begin{cases} \forall a < 2, & n^a \ll u_n \\ \forall a \geq 2, & u_n \ll n^a \end{cases}$$

2. Construct a sequence $\{v_n\}_n$ such that

$$\begin{cases} \forall a \leq 2, & n^a \ll v_n \\ \forall a > 2, & v_n \ll n^a \end{cases}$$

Recall the definitions

1. Let f be a bounded, continuous function on $[c, \infty)$. How do we define the improper integral
$$\int_c^\infty f(x)dx\,?$$

2. Let f be a continuous function on $(a, b]$. How do we define the improper integral
$$\int_a^b f(x)dx\,?$$

Computation

Calculate, using the definition of improper integral

$$\int_1^{\infty} \frac{1}{x^2 + x} dx$$

Warm-up

Is there a difference between:

$$\lim_{x\to\infty} x - \lim_{x\to\infty} x$$

and

$$\lim_{x\to\infty} (x - x)?$$

A “simple” integral

What is $\displaystyle\int_{-1}^{1} \frac{1}{x}\, dx$?

A "simple" integral

What is $\int_{-1}^{1} \frac{1}{x}\, dx$?

1. $\int_{-1}^{1} \frac{1}{x}\, dx = (\ln|x|)\Big|_{-1}^{1} = \ln|1| - \ln|-1| = 0$
2. $\int_{-1}^{1} \frac{1}{x}\, dx = 0$ because $f(x) = \frac{1}{x}$ is an odd function.
3. $\int_{-1}^{1} \frac{1}{x}\, dx$ is divergent.

The most important improper integrals

Use the definition of improper integral to determine for which values of $p \in \mathbb{R}$ each of the following improper integrals converges.

1. $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx$
2. $\displaystyle\int_0^1 \frac{1}{x^p}\,dx$
3. $\displaystyle\int_0^\infty \frac{1}{x^p}\,dx$

Computation

Does $\int_0^\infty \frac{1}{x^2-3x+2}$ converge or diverge?

1. Write down the definition of this improper integral.
2. Compute the improper integral from the definition.

A simple BCT application

We want to determine whether $\int_1^\infty \frac{1}{x + e^x}\,dx$ is convergent or divergent.

We can try at least two comparisons:

1. Compare $\frac{1}{x}$ and $\frac{1}{x + e^x}$.
2. Compare $\frac{1}{e^x}$ and $\frac{1}{x + e^x}$.

Try both. What can you conclude from each one of them?

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True or False

Let $a \in \mathbb{R}$.

Let f and g be continuous functions on $[a, \infty)$.

Assume that $\boxed{\forall x \geq a, \quad 0 \leq f(x) \leq g(x)}$.

What can we conclude?

1. IF $\int_a^\infty f(x)dx$ is convergent, THEN $\int_a^\infty g(x)dx$ is convergent.
2. IF $\int_a^\infty f(x)dx = \infty$, THEN $\int_a^\infty g(x)dx = \infty$.
3. IF $\int_a^\infty g(x)dx$ is convergent, THEN $\int_a^\infty f(x)dx$ is convergent.
4. IF $\int_a^\infty g(x)dx = \infty$, THEN $\int_a^\infty f(x)dx = \infty$.

True or False - Part II

Let $a \in \mathbb{R}$.

Let f and g be continuous functions on $[a, \infty)$.

Assume that $\boxed{\forall x \geq a, \quad f(x) \leq g(x)}$.

What can we conclude?

1. IF $\int_a^\infty f(x)dx$ is convergent, THEN $\int_a^\infty g(x)dx$ is convergent.
2. IF $\int_a^\infty f(x)dx = \infty$, THEN $\int_a^\infty g(x)dx = \infty$.
3. IF $\int_a^\infty g(x)dx$ is convergent, THEN $\int_a^\infty f(x)dx$ is convergent.
4. IF $\int_a^\infty g(x)dx = \infty$, THEN $\int_a^\infty f(x)dx = \infty$.

True or False - Part III

Let $a \in \mathbb{R}$.
Let f and g be continuous functions on $[a, \infty)$.
Assume that $\boxed{\exists M \geq a \text{ s.t. } \forall x \geq M, \quad 0 \geq f(x) \geq g(x)}$.
What can we conclude?

1. IF $\int_a^\infty f(x)dx$ is convergent, THEN $\int_a^\infty g(x)dx$ is convergent.
2. IF $\int_a^\infty f(x)dx = -\infty$, THEN $\int_a^\infty g(x)dx = -\infty$.
3. IF $\int_a^\infty g(x)dx$ is convergent, THEN $\int_a^\infty f(x)dx$ is convergent.
4. IF $\int_a^\infty g(x)dx = -\infty$, THEN $\int_a^\infty f(x)dx = -\infty$.

BCT calculations

Use the BCT to determine whether each of the following is convergent or divergent

1. $$\int_1^\infty \frac{1+\cos^2 x}{x^{2/3}}\,dx$$
2. $$\int_1^\infty \frac{1+\cos^2 x}{x^{4/3}}\,dx$$
3. $$\int_2^\infty \frac{(\ln x)^{10}}{x^2}\,dx$$
4. $$\int_2^\infty \frac{x^2+1}{x^4-2}\,dx$$
5. $$\int_0^\infty e^{-x^2}\,dx$$
6. $$\int_0^\infty \frac{\arctan x^2}{1+e^x}\,dx$$

What can you conclude?

Let $a \in \mathbb{R}$. Let f be a continuous, positive function on $[a, \infty)$.

In each of the following cases, what can you conclude about $\int_a^{\infty} f(x)dx$?

Is it convergent, divergent, or we do not know?

1. $\forall b \geq a, \ \exists M \in \mathbb{R}$ s.t. $\int_a^b f(x)dx \leq M$.
2. $\exists M \in \mathbb{R}$ s.t. $\forall b \geq a, \ \int_a^b f(x)dx \leq M$.
3. $\exists M > 0$ s.t. $\forall x \geq a, \ f(x) \leq M$.
4. $\exists M > 0$ s.t. $\forall x \geq a, \ f(x) \geq M$.