Today's topics and news

- Topic: Big Theorem
- **Homework:** Watch videos 12.1 - 12.8 for Tuesday (12.2, 12.3, and 12.6 are supplementary) and 12.9 - 12.10 for Wednesday.
- Have a good reading week!

A suspicious calculation – What is wrong?

The sequence $\{a_n\}_{n=0}^{\infty}$ defined by

$$\begin{cases} & a_0 = 1 \\ \forall n \in N, & a_{n+1} = 1 - a_n \end{cases}$$

has limit $1/2$.

Proof.

- Let $L = \lim_{n\to\infty} a_n$.
- $a_{n+1} = 1 - a_n$
- $\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} [1 - a_n]$
- $L = 1 - L$
- $L = 1/2$.

□

Application of MCT

We will check the convergence of the sequence $\{a_n\}_{n=0}^{\infty}$ defined recursively as follows:
$a_0 = \sqrt{2}$
$a_{n+1} = \sqrt{2 + a_n}\ \forall n \in \mathbb{N}$

Rough work:
1. Guess whether a_n is increasing or decreasing. Don't try to prove it yet.
2. If a_n does converge to some a, taking limits of the recursive relation, what must a be? (**Keep in mind this is completely hypothetical as you have not yet proved that a_n converges.**)
3. Guess an upper bound and a lower bound for a_n using 1 and 2, which is needed for MCT to work?

Proofs:
4. Prove your guess in 1.
5. Prove your bound in 3.
6. Does a_n converge? If so what does it converge to?

Much less than and the Big Theorem

Much less than

Given positive sequences a_n and b_n,
we say $a_n \ll b_n$ iff $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$

The Big Theorem

$$ln(n) \ll n^a \ll c^n \ll n! \ll n^n$$

for every $a > 0, c > 1$.

Calculations

1. $\lim_{n\to\infty} \frac{n! + 2e^n}{3n! + 4e^n}$

2. $\lim_{n\to\infty} \frac{2^n + (2n)^2}{2^{n+1} + n^2}$

3. $\lim_{n\to\infty} \frac{5n^5 + 5^n + 5n!}{n^n}$

Much less than – True or False

Let $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ be positive sequences.

1. IF $a_n \ll b_n$, THEN $\forall m \in \mathbb{N},\ a_m < b_m$.
2. IF $a_n \ll b_n$, THEN $\exists n_0 \in \mathbb{N}$ s.t. $\forall m \in \mathbb{N},\ m \geq n_0 \implies a_m < b_m$.
3. IF $\forall m \in \mathbb{N},\ a_m < b_m$, THEN $a_n \ll b_n$.
4. IF $\exists n_0 \in \mathbb{N}$ s.t. $\forall m \in \mathbb{N},\ m \geq n_0 \implies a_m < b_m$, THEN $a_n \ll b_n$.

Refining the Big Theorem

We know from the Big Theorem that

The Big Theorem

$$ln(n) \ll n^a \ll c^n \ll n! \ll n^n$$

for every $a > 0, c > 1$.

There are actually lots of sequences in between each of these.

Exercise: Construct new sequencse $\{u_n\}$, $\{v_n\}$, $\{w_n\}$, $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ such that, for every $a > 0$ and for every $c > 1$,

$$u_n \ll ln(n) \ll v_n \ll n^a \ll w_n \ll c^n \ll x_n \ll n! \ll y_n \ll n^n \ll z_n$$

Refining the Big Theorem

1. Construct a sequence $\{u_n\}_n$ such that

$$\begin{cases} \forall a < 2, & n^a \ll u_n \\ \forall a \geq 2, & u_n \ll n^a \end{cases}$$

2. Construct a sequence $\{v_n\}_n$ such that

$$\begin{cases} \forall a \leq 2, & n^a \ll v_n \\ \forall a > 2, & v_n \ll n^a \end{cases}$$