Today's topics and news

- Topic: More on sequences, MCT
- **Homework:** Watch videos 11.7 and 11.8 for Wednesday.

Definition of limit of a sequence

Let $\{a_n\}_{n=0}^{\infty}$ be a sequence. Let $L \in \mathbb{R}$.
Which statements are equivalent to "$\{a_n\}_{n=0}^{\infty} \longrightarrow L$"?

1. $\forall \varepsilon > 0,\ \exists n_0 \in \mathbb{N}$ s.t. $\forall n \in \mathbb{N},\quad n \geq n_0 \implies |L - a_n| < \varepsilon.$
2. $\forall \varepsilon > 0,\ \exists n_0 \in \mathbb{N}$ s.t. $\forall n \in \mathbb{N},\quad n > n_0 \implies |L - a_n| < \varepsilon.$
3. $\forall \varepsilon > 0,\ \exists n_0 \in \mathbb{R}$ s.t. $\forall n \in \mathbb{N},\quad n \geq n_0 \implies |L - a_n| < \varepsilon.$
4. $\forall \varepsilon > 0,\ \exists n_0 \in \mathbb{N}$ s.t. $\forall n \in \mathbb{N},\quad n \geq n_0 \implies |L - a_n| \leq \varepsilon.$
5. $\forall \varepsilon \in (0,1),\ \exists n_0 \in \mathbb{N}$ s.t. $\forall n \in \mathbb{N},\quad n \geq n_0 \implies |L - a_n| < \varepsilon.$
6. $\forall \varepsilon > 0,\ \exists n_0 \in \mathbb{N}$ s.t. $\forall n \in \mathbb{N},\quad n \geq n_0 \implies |L - a_n| < \dfrac{1}{\varepsilon}.$
7. $\forall k \in \mathbb{Z}^+ > 0,\ \exists n_0 \in \mathbb{N}$ s.t. $\forall n \in \mathbb{N},\quad n \geq n_0 \implies |L - a_n| < k.$
8. $\forall k \in \mathbb{Z}^+ > 0,\ \exists n_0 \in \mathbb{N}$ s.t. $\forall n \in \mathbb{N},\quad n \geq n_0 \implies |L - a_n| < \dfrac{1}{k}.$

Definition of limit of a sequence (continued)

Let $\{a_n\}_{n=0}^{\infty}$ be a sequence. Let $L \in \mathbb{R}$.
Which statements are equivalent to "$\{a_n\}_{n=0}^{\infty} \longrightarrow L$"?

9. $\forall \varepsilon > 0$, the interval $(L - \varepsilon, L + \varepsilon)$ contains all the elements of the sequence, except the first few.
10. $\forall \varepsilon > 0$, the interval $(L - \varepsilon, L + \varepsilon)$ contains all the elements of the sequence, except finitely many.
11. Every interval that contains L must contain all but finitely many of the terms of the sequence.
12. Every open interval that contains L must contain all but but finitely many of the terms of the sequence.

Convergence and divergence

Let $\{a_n\}_{n=0}^{\infty}$ be a sequence.
Write the formal definition of the following concepts:

1. $\{a_n\}_{n=0}^{\infty}$ is convergent.
2. $\{a_n\}_{n=0}^{\infty}$ is divergent.
3. $\{a_n\}_{n=0}^{\infty}$ is divergent to ∞.

Properties

All the usual properties you know for limits of functions more or less applies to limits of sequences. In particular, (provided the RHS limits exist):

1. $\lim_{n\to\infty}(a_n + kb_n) = \lim_{n\to\infty} a_n + k \lim_{n\to\infty} b_n$.
2. $\lim_{n\to\infty}(a_n b_n) = \lim_{n\to\infty} a_n \lim_{n\to\infty} b_n$.
3. $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$ provided $\lim_{n\to\infty} b_n \neq 0$.
4. If f is continuous, then $\lim_{n\to\infty} f(a_n) = f(\lim_{n\to\infty} a_n)$.
5. If $a_n \leq b_n \leq c_n$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n$, then $\lim_{n\to\infty} b_n$ exists and equal the two limits.

The proofs of all of these are very similar to the corresponding proofs in the function case. Homework: Try proving a couple.

Examples

Construct 8 examples of sequences.
If any of them is impossible, cite a theorem to justify it.

		convergent	divergent
monotonic	bounded	???	???
	unbounded	???	???
not monotonic	bounded	???	???
	unbounded	???	???

Cauchy sequences

Cauchy sequence

A sequence $\{a_n\}_{n=0}^{\infty}$ is a Cauchy sequence iff

$\forall \epsilon > 0, \exists N \in \mathbb{N}$ s.t. $\forall m, n \in \mathbb{N}$,
if $m, n \geq N$ then $|a_m - a_n| < \epsilon$.

Prove that if a sequences $\{a_n\}_{n=0}^{\infty}$ converges then it is a Cauchy sequences.

It is also true that if a sequence is Cauchy then it is must converge. However, this is beyond the scope of this course. You do not need to know the definition of Cauchy sequences for this course.

Warm up - True or false

1. Every convergent sequence is eventually monotone, that is, eventually increasing or decreasing.

2. If $\lim_{n\to\infty} a_n = L$ then $\lim_{n\to\infty} a_{n^3} = L$.

3. If $\lim_{n\to\infty} a_{2n} = L$ then $\lim_{n\to\infty} a_n = L$.

4. If a sequence diverges and is increasing, then there exists $n \in \mathbb{N}$ such that $a_n > 100$.

Warm up - True or false

1. If a sequence is non-decreasing and non-increasing, then it is convergent.

2. If a sequence is not decreasing and not increasing, then it is convergent.

3. If a sequence is increasing and decreasing, then it is convergent.

Warm up

1. Suppose a_n converges and every number in the sequence is an integer. What can you say about the sequence?
2. You learned about "monotone" and "eventually monotone" sequences in the videos. Why do we not need to define what it means to be an "eventually bounded sequence"?

A suspicious calculation – What is wrong?

Claim: The sequence $\{a_n\}_{n=0}^{\infty}$ defined by

$$\begin{cases} & a_0 = 1 \\ \forall n \in N, & a_{n+1} = 1 - a_n \end{cases}$$

has limit $1/2$.

Proof.

- Let $L = \lim_{n\to\infty} a_n$.
- $a_{n+1} = 1 - a_n$
- $\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} [1 - a_n]$
- $L = 1 - L$
- $L = 1/2$.

□

Application of MCT

We will check the convergence of the sequence $\{a_n\}_{n=0}^{\infty}$ defined recursively as follows:
$a_0 = \sqrt{2}$
$a_{n+1} = \sqrt{2 + a_n} \; \forall n \in \mathbb{N}$

Rough work:
1. Guess whether a_n is increasing or decreasing. Don't try to prove it yet.
2. If a_n does converge to some a, taking limits of the recursive relation, what must a be? (**Keep in mind this is completely hypothetical as you have not yet proved that a_n converges.**)
3. Guess an upper bound and a lower bound for a_n using 1 and 2, which is needed for MCT to work?

Proofs:
4. Prove your guess in 1.
5. Prove your bound in 3.
6. Does a_n converge? If so what does it converge to?