Today's topics and news

- Topic: Sup, inf, and defintion of the integral

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- **Homework for Tuesday:** Watch videos 7.8 and 7.9.

Warm up: suprema and infima

Find the supremum, infimum, maximum, and minimum of the following sets (if they exist):

1. $A = [-1, 5)$
2. $B = (-\infty, 0) \cup (3, \infty)$
3. $C = \{2, 3, 4\}$
4. $D = \left\{ \dfrac{1}{n} \ : \ n \in \mathbb{Z}, n > 0 \right\}$
5. $E = \left\{ \dfrac{(-1)^n}{n} \ : \ n \in \mathbb{Z}, n > 0 \right\}$
6. $F = \{2^n \ : \ n \in \mathbb{Z}\}$

Empty set

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Recall:

Let $A \subseteq \mathbb{R}$. Let $a \in \mathbb{R}$.

- a is an **upper bound** of A means $\forall x \in A, x \leq a$.
- a is the **least upper bound** (lub) or **supremum** (sup) of A means
  - a is an upper bound of A, and
  - there are no smaller upper bounds.

sup and inf existence theorem

sup existence theorem

Let $A \subseteq \mathbb{R}$,

A has a supremum iff A is bounded above and non-empty.

Equivalent definitions of supremum

Assume u is an upper bound of the set A, which of the following statements are equivalent to $u = \sup(A)$?

1. $\forall v \leq u$, v is not an upper bound of A.
2. $\forall v < u$, v is not an upper bound of A.
3. $\forall v < u$, $\exists x \in A$ s.t. $v < x$.
4. $\forall v < u$, $\exists x \in A$ s.t. $v \leq x$.
5. $\forall v < u$, $\exists x \in A$ s.t. $v < x \leq u$.
6. $\forall v < u$, $\exists x \in A$ s.t. $v < x < u$.
7. $\forall \epsilon > 0$, $\exists x \in A$ s.t. $u - \epsilon < x \leq u$.
8. $\forall \epsilon > 0$, $\exists x \in A$ s.t. $u - \epsilon < x < u$.

Sup and inf proof

Let $A = [0, 1)$

Prove $\inf(A) = 0$:

1. Check 0 a lower bound.
2. Suppose l is another lower bound of A, why does 0 have to be larger?

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Prove $\sup(A) = 1$:
1. Check 1 an upper bound?
2. Suppose u is another upper bound of A, and assume it's less than 1, come up with a number in A (using u) which is for sure larger than u, therefore contradicting the fact that u is an upper bound.

Partitions

Which of the following are partitions of $[0, 2]$?

1. $[0, 2]$
2. $(0, 2)$
3. $\{0, 2\}$
4. $\{1, 2\}$
5. $\{0, 1, 1.5, 2\}$

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A **partition** of $[a, b]$ is **expressed** as a finite set S where $S \subseteq [a, b]$ and $a,\ b \in S$. It should be **thought of** as a way of dividing up the interval $[a, b]$, where you divide $[a, b]$ at all elements of S.

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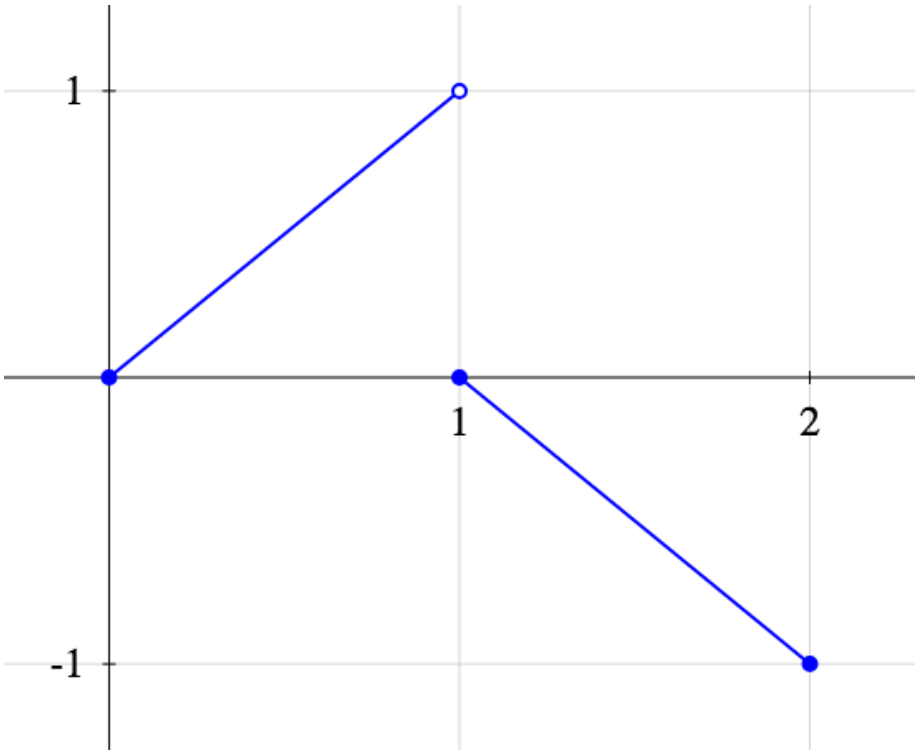
Definition of upper sum

Given a **bounded function** f on $[a, b]$ **and** a **partition** P, there are two ways to estimate the"signed area under the curve of f". These are called upper sum $U_P(f)$ and the lower sum $L_P(f)$. As you will see in the next slide, these estimates **depend** on the partition.

Given a partition $\{a = x_0 < x_1 < ...x_n = b\}$ of $[a, b]$ and a function f. Define $U_P(f)$.

Computing $U_P(f)$



Compute $U_P(f)$ for the following partitions:

1. $\{0, 2\}$
2. $\{0, 0.5, 1.5, 2\}$

Upper and lower integrals

We see that given a bounded function f on $[a, b]$ and a partition P of $[a, b]$, we can produce two estimates for the area under f between a and b, one of which is an underestimate and one of which is an overestimate.

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1. The upper integral $\overline{I_a^b}(f) :=$
2. The lower integral $\underline{I_a^b}(f) :=$

Integrability

1. The upper integral $\overline{I_a^b}(f) := \inf(\{U_P(f) : \text{P is a partition of } [a, b]\})$
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These are the best possible candidates for area under f and there's no preference for one over the other. That's why if they are not equal (i.e. $\overline{I_a^b}(f) > \underline{I_a^b}(f)$), we don't have a good notion of area and we say the function is not integrable. And if they are equal, then the function is **integrable** and

$$\int_a^b f(x)dx = \overline{I_a^b}(f) = \underline{I_a^b}(f).$$

Lower and upper sums

Let f be a **decreasing**, bounded function on $[a, b]$.
Let $P = \{x_0, x_1, \ldots, x_N\}$ be a partition of $[a, b]$

Which one (or ones) is a valid equation for $L_P(f)$? For $U_f(P)$?

1. $\displaystyle\sum_{i=0}^{N} \Delta x_i \, f(x_i)$
2. $\displaystyle\sum_{i=1}^{N} \Delta x_i \, f(x_i)$
3. $\displaystyle\sum_{i=0}^{N-1} \Delta x_i \, f(x_i)$
4. $\displaystyle\sum_{i=1}^{N} \Delta x_i \, f(x_{i+1})$
5. $\displaystyle\sum_{i=1}^{N} \Delta x_i \, f(x_{i-1})$
6. $\displaystyle\sum_{i=0}^{N-1} \Delta x_{i+1} \, f(x_i)$

Recall: $\Delta x_i = x_i - x_{i-1}$.

A very tricky question

Let f be a bounded function on $[a, b]$.
Which statement is true?

1. There exists a partition P of $[a, b]$ such that
$$\underline{I_a^b}(f) = L_P(f) \qquad \text{and} \qquad \overline{I_a^b}(f) = U_P(f).$$

2. There exist partitions P and Q of $[a, b]$ such that
$$\underline{I_a^b}(f) = L_P(f) \qquad \text{and} \qquad \overline{I_a^b}(f) = U_Q(f).$$

The "ε–characterization" of integrability - First proof

You are going to prove

Claim

Let f be a bounded function on $[a, b]$.

- IF f is integrable on $[a, b]$
- THEN "$\forall \varepsilon > 0$, $\exists$ a partition P of $[a, b]$, s.t. $U_P(f) - L_P(f) < \varepsilon$".

1. Recall the definition of "f is integrable on $[a, b]$".
2. Write down the structure of the proof.
3. Fix $\varepsilon > 0$. Show there is a partition P s.t. $U_P(f) < \overline{I_a^b}(f) + \frac{\varepsilon}{2}$. Why?
4. Assume f is integrable on $[a, b]$. Fix $\varepsilon > 0$.
   Show there are partitions P_1 and P_2 s.t. $U_{P_1}(f) - L_{P_2}(f) < \varepsilon$.
5. Using P_1 and P_2 from the previous step, construct a partition P such that $U_P(f) - L_P(f) < \varepsilon$.
6. Write down a proof for the claim.

The "ε–characterization" of integrability - Second proof

You are going to prove

Claim

Let f be a bounded function on $[a, b]$.

- IF "$\forall \varepsilon > 0$, $\exists$ a partition P of $[a, b]$, s.t. $U_P(f) - L_P(f) < \varepsilon$".
- THEN f is integrable on $[a, b]$

1. Recall the definition of "f is integrable on $[a, b]$".
2. For any partition P, order the quantities $U_P(f)$, $L_P(f)$, $\overline{I_a^b}(f)$, $\underline{I_a^b}(f)$.
3. For any partition P, order the quantities $U_P(f) - L_P(f)$, $\overline{I_a^b}(f) - \underline{I_a^b}(f)$, and 0.
4. Write down a proof for the claim.

The "ε–characterization" of integrability

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Let f be a bounded function on $[a, b]$.

f is integrable on $[a, b]$

IFF

$\forall \varepsilon > 0$, $\exists$ a partition P of $[a, b]$, s.t. $U_P(f) - L_P(f) < \varepsilon$.