MAT 137
Tutorial #18– Taylor series
August 8/12, 2019

There are four main power series we know. If you have not learned them in lecture yet, you will soon. For now, just accept them. They are:

$$\begin{aligned}
e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots && \text{for all } x \\
\sin x &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots && \text{for all } x \\
\cos x &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots && \text{for all } x \\
\frac{1}{1-x} &= \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \dots && \text{for } |x| < 1
\end{aligned}$$

In Questions 1-3 you will practice how to manipulate these four functions to write many more functions as power series. In Question 4 you will use this to actually calculate the sum of a lot of series.

1. Compute the Maclaurin series of the following functions by performing algebraic manipulations of the main four Taylor series. Indicate the domain where the expansion is valid.

   Hint: As an example, for Question 1a, use the substitution $u = -x$ and use the Maclaurin expansion for $\frac{1}{1-u}$. Similar tricks work for many of the others.

   (a) $f(x) = \frac{1}{1+x}$

   (b) $f(x) = \frac{1}{1+x^2}$

   (c) $f(x) = x^2 e^x$

   (d) $f(x) = \cos x^4$

   (e) $f(x) = \sin^2 x$

   (f) $f(x) = \frac{e^x - e^{-x}}{2}$

   (g) $f(x) = \frac{1}{x^2 - 3x + 2}$

   (h) $f(x) = \frac{\sin x}{x}$

   Notes: Question 1h actually means $f(x) = \begin{cases} \frac{\sin x}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$

   For Question 1g, you can use partial fraction decomposition.

2. Compute the Maclaurin series of the following functions using differentiation or integration of other known series. Indicate the domain where the expansion is valid.

   (a) $\ln(1+x)$
   (b) $\arctan x$
   (c) $f(x) = \int_0^x e^{-t^2}\,dt$

   Hint: As an example, for Question 2a, notice that you already know the Maclaurin expansion for $\frac{d}{dx}\ln(1+x) = \frac{1}{1+x}$. Then integrate that series term by term. Pay attention to the constant of integration.

3. Compute the Taylor series of the following functions about $x = a$ for the given value of a. Indicate the domain where the expansion is valid.

   (a) $f(x) = e^x$ about $a = 1$.
   (b) $f(x) = \frac{1}{1-x}$ about $a = 3$.
   (c) $f(x) = \sin x$ about $a = \pi/4$.

   Hint: You can always use the substitution $u = x - a$ and try to turn the resulting function into something you recognize.

4. Compute the value of the following series:

   (a) $\sum_{n=1}^{\infty} nx^n$
   (b) $\sum_{n=0}^{\infty} \frac{n^2}{2^n}$
   (c) $\sum_{n=0}^{\infty} \frac{x^{n+1}}{n!}$
   (d) $\sum_{n=0}^{\infty} \frac{(n+1)x^n}{n!}$
   (e) $\sum_{n=0}^{\infty} \frac{x^n}{(n+3)n!}$
   (f) $\sum_{n=0}^{\infty} (-1)^n \frac{(n+1)x^{2n+1}}{(2n+1)!}$
   (g) $\sum_{n=0}^{\infty} \frac{x^n}{(n+1)(n+2)}$
   (h) $\sum_{n=2}^{\infty} \frac{(4n^2+8n+3)\,2^n}{n!}$

   Hint: This is what you did in Question 1, but backwards. Try to start with a series that looks "similar" to the one you want, but whose value you know. Then manipulate it algebraically and/or take derivatives and/or integrals until you get the series you want.

   For example, for Question 4a, notice that $\sum_{n=1}^{\infty} nx^{n-1} = \frac{d}{dx}\sum_{n=0}^{\infty} x^n$