TEST 2 SOLUTIONS

(1) (a) Suppose E is a subset of a metric space (X,d). The diameter of E is defined as $\operatorname{diam} E = \sup\{d(x,y) : x, y \in E\}$. Suppose X is complete and (E_n) is a decreasing sequence of closed non-empty subsets of X such that $\lim_{n\to\infty} \operatorname{diam} E_n = 0$. Prove that $\bigcap_{n\geq 1} E_n$ is non-empty.

Pick a point $x_n \in E_n$ for each n.
Claim. The sequence (x_n) is Cauchy.
Certainly, given $\epsilon > 0$, there exists N such that $\operatorname{diam} E_N < \epsilon$. Then for $m, n \geq N$, we have $x_n, x_m \in E_N$, and so

$$d(x_n, x_m) \leq \operatorname{diam} E_N < \epsilon.$$

Since X is complete, the sequence (x_n) converges. Let $x \in X$ be its limit.
For each n, we have $(x_m)_{m\geq n}$ is a sequence contained in E_n, and this sequence converges to x. Since E_n is closed, we must have $x \in E_n$.
Thus, $x \in E_n$ for all n, so $x \in \bigcap_{n\geq 1} E_n$.

(b) Give an example of a complete metric space X and a decreasing sequence (E_n) of closed subsets of X whose intersection is empty.

Let $X = \{1, 2, 3, \dots\}$ and let d be the discrete metric, that is,

$$d(x,y) = \begin{cases} 1, & \text{if } x \neq y, \\ 0, & \text{if } x = y. \end{cases}$$

This metric is complete, and with this metric, every subset of X is closed.
Let $E_n = \{n, n+1, n+2, \dots\}$. Since every subset of X is closed, in particular E_n is closed. Also, we clearly have $E_n \supset E_{n+1}$ so that the sequence is decreasing. Finally, it is obvious that $\bigcap_{n\geq 1} E_n = \varnothing$.

(2) (a) Suppose (X,d) is a complete metric space and $f : X \to X$ is a contraction mapping, that is there exists $r \in [0,1)$ such that

$$d(f(x), f(y)) \leq r d(x,y) \ \forall\, x, y \in X.$$

Let a be any point in X and define a sequence (x_n) recursively by $x_1 = a$ and $x_{n+1} = f(x_n)$. Show that (x_n) converges to a limit $x_0 \in X$ and that x_0 is a fixed point of f $(f(x_0) = x_0)$. Prove that f has only one fixed point.

To show that (x_n) converges, we shall show that it is Cauchy. Certainly, we have

$$d(x_{n+1}, x_{n+2}) = d(f(x_n), f(x_{n+1})) \leq rd(x_n, x_{n+1}),$$

so that by induction, $d(x_n, x_{n+1}) \leq r^{n-1} d(x_1, x_2)$.

From this we see that, for $m \geq n$,

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\ &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots \\ &\leq r^{n-1} d(x_1, x_2) + r^n d(x_1, x_2) + \cdots \\ &= \frac{r^{n-1}}{1-r} d(x_1, x_2) \to 0 \end{aligned}$$

as $n \to \infty$. This shows that the sequence is Cauchy.

Hence, it has a limit x_0. We have

$$x_0 = \lim_{n\to\infty} x_n = \lim_{n\to\infty} f(x_{n-1}) = f(x_0),$$

by continuity of f. This shows that x_0 is a fixed point.

If y is another fixed point then

$$d(x_0, y) = d(f(x_0), f(y)) \leq rd(x_0, y),$$

and since $r < 1$, this can only happen if $d(x_0, y) = 0$. This shows that $y = x_0$, and hence there is only one fixed point.

(b) Define a sequence (x_n) of real numbers recursively by $x_1 = 0$ and $x_{n+1} = \frac{x_n^2}{3} - 1$. Prove that (x_n) is convergent and find its limit.

Let $f : [-1, 1] \to \mathbb{R}$ be defined by

$$f(x) = \frac{x^2}{3} - 1.$$

Then

$$f'(x) = \frac{2x}{3}, \tag{*}$$

so that on $[-1, 1]$, $|f'(x)| < \frac{2}{3}$. It follows from the Mean Value Theorem that, for $x, y \in [-1, 1]$,

$$|f(x) - f(y)| \leq \frac{2}{3}|x - y|.$$

From (*), we also see that f is increasing on $[-1, 0]$ and decreasing on $[0, 1]$. Thus, the minimum value of f is $f(0) = -1$ while its maximum value is

$$\max\{f(-1), f(1)\} = \max\{-\frac{2}{3}, -\frac{2}{3}\} = -\frac{2}{3}.$$

In particular, we see that the range of f is contained in $[-1, 1]$.

We can thus apply part (a) with $X = [-1, 1]$ and $r = \frac{2}{3}$. Note that the sequence (x_n) satisfies $f(x_n) = x_{n+1}$, so part (a) shows that the sequence converges, and its limit x_0 satisfies $f(x_0) = x_0$. Explicitly, this says

$$\frac{x_0^2}{3} - 1 = x_0$$
$$x_0^2 - 3x_0 - 3 = 0.$$

Using quadratic formula, we derive that

$$x_0 = \frac{3 \pm \sqrt{(-3)^2 - 4(1)(-3)}}{2} = \frac{3 \pm \sqrt{21}}{2}.$$

We also know that $x_0 \in [-1, 1]$, which rules out $\frac{3+\sqrt{21}}{2}$ (which is > 1). Hence, $x_0 = \frac{3-\sqrt{21}}{2}$.

(3) Determine convergence or divergence for the series

(a) $\sum_{n=1}^{\infty} \frac{(2n)!}{n^{2n}}$.

Let us use the ratio test. If we label the terms a_n then we have

$$\begin{aligned} \frac{a_n}{a_{n+1}} &= \frac{(2n)!}{n^{2n}} \div \frac{(2n+2)!}{(n+1)^{2n+2}} \\ &= \frac{1}{(2n+2)(2n+1)} \left(\frac{n+1}{n}\right)^{2n} (n+1)^2 \\ &= \left(\left(1 + \frac{1}{n}\right)^n\right)^2 \frac{(n+1)^2}{(2n+2)(2n+1)} \\ &\to \frac{e^2}{4} > 1 \end{aligned}$$

It follows that the series diverges.

(b) $\sum_{n=3}^{\infty} \frac{\log(\log n)}{n(\log n)^2}$.

Let us use the integral test. Consider the function $f(t) = \frac{\log(\log t)}{t(\log t)^2}$. We have

$$\begin{aligned} f'(t) &= \frac{1}{t(\log t)^2(\log t)t} - \frac{\log(\log t)}{t^2(\log t)^2} - 2\frac{\log(\log t)}{t(\log t)^3 t} \\ &= \frac{1 - (\log t)\log(\log t) - 2\log(\log t)}{t^2(\log t)^3}. \end{aligned}$$

From this it is clear that $f(t) < 0$ for t sufficiently large, so that f is eventually decreasing. Hence, we may apply the integral test: the series $\sum_{n=3}^{\infty} f(n)$ converges if and only if the integral $\int_3^{\infty} f(t)\,dt$ converges.

To evaluate the integral, we shall use the substitution $u = \log t$, so that $du = \frac{dt}{t}$. We have

$$\int_3^\infty \frac{\log(\log t)}{t(\log t)^2}\,dt = \int_{\log 3}^\infty \frac{\log u}{u^2}\,du.$$

Next, we know that $\log u \leq \sqrt{u}$ for $u \geq \log 3$, so that

$$\begin{aligned}\int_{\log 3}^\infty \frac{\log u}{u^2}\,du &\leq \int_{\log 3}^\infty \frac{\sqrt{u}}{u^2}\,du \\ &= \int_{\log 3}^\infty u^{-3/2}\,du \\ &= \frac{-1}{2}u^{-1/2}\Big|_{\log 3}^\infty < \infty.\end{aligned}$$

Hence, the integral $\int_3^\infty \frac{\log(\log t)}{t(\log t)^2}\,dt$ converges and therefore so does the series $\sum_{n=3}^\infty \frac{\log(\log n)}{n(\log n)^2}$.

(4) (a) Complete the following statement of the fundamental theorem of calculus: Suppose that f is continuous on $[a, b]$, $c \in [a, b]$ and define $F(x) = \int_c^x f(t)\,dt$. Then ...

Then F is differentiable and $F'(x) = f(x)$ for $x \in (a, b)$.

(b) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is differentiable everywhere, $f'(x) \neq 0$ for $x \neq 0$ and that f satisfies the equation

$$\int_0^{f(x)} e^{t^3}\,dt = (x^2 + 1)f(x) - 2\int_0^x tf(t)\,dt$$

for all $x \in \mathbb{R}$. Determine the function f. Justify your reasoning.

Let us differentiate both sides of the given equation. To do this, we use the fundamental theorem of calculus on both sides. We can easily see that the functions inside the integrals are continuous.

$$\begin{aligned}e^{f(x)^3}f'(x) &= 2xf(x) + (x^2 + 1)f'(x) - 2xf(x) \\ &= (x^2 + 1)f'(x).\end{aligned}$$

For $x \neq 0$, we have $f'(x) \neq 0$ and so

$$\begin{aligned}e^{f(x)^3} &= x^2 + 1 \\ f(x) &= \sqrt[3]{\log(x^2 + 1)}.\end{aligned}$$

Since f is continuous, this equation must hold for $x = 0$ also.