Math 337 08-09 Test 1 Feb. 25

Name:____

Note: this test consists of 4 pages.

(1) In this question E always denotes a subset of $\mathbb{R}^d$.

(a) (7 points) Define what it means for E to be open.

(b) (7 points) Define what it means for E to be closed.

(c) (11 points) Show that E is closed if and only if $E' = \mathbb{R}^d \backslash E$ is open.

(d) (7 points) Define what is meant by a Cauchy sequence in $\mathbb{R}^d$.

(e) (7 points) State the completeness property of $\mathbb{R}^d$.

(f) (11 points) Define what it means for $E \subset \mathbb{R}^d$ to be compact and give an equivalent characterization of compactness of E.

(2) Suppose that $f : \mathbb{R}^d \to \mathbb{R}^d$ and there is a constant $c \in (0, 1)$ such that $\|f(x) - f(y)\| \le c\|x - y\|$ for all $x, y \in \mathbb{R}^d$. Let $x_0 \in \mathbb{R}^d$ be an arbitrary point in $\mathbb{R}^d$, let $x_1 = f(x_0),\ x_2 = f(x_1)$ and in general $x_{n+1} = f(x_n)$. Prove the following.

(a) (7 points) f is continuous everywhere.

(b) (10 points) $\|x_{n+1} - x_n\| \le c^n r$, where $r = \|x_1 - x_0\|$.

(c) (13 points) For each $m > 0$ we have
$\|x_{n+m} - x_n\| \le c^n r + c^{n+1} r + \ldots + c^{n+m-1} r \le c^n \frac{r}{1-c}$.
(Use (b) and the triangle inequality.)

(d) (13 points) Deduce that $\{x_n\}$ is Cauchy and hence converges to a limit y. Show that y is a fixed point of f, that is $f(y) = y$, and that f has exactly one fixed point.