Math 337 Winter 2010, Problem set 6 Solutions to selected problems

§8.1 B. First, since $nt^n \to 0$ for $t \in [0,1)$ we have $f_n(x) \to 0$ for all $x \in [0,1]$. Calculus reveals that on $[0,1]$ f_n has a maximum at

$$x_n = \frac{1}{\sqrt{2n+1}} \text{and } f_n(x_n) = \frac{n}{\sqrt{2n+1}}(\frac{2n}{2n+1})^n.$$

Since the second factor goes to $e^{\frac{-1}{2}}$ it follows that $f_n(x_n) \to \infty$ so $\|f_n\|_\infty \to \infty$ and the convergence is certainly not uniform.

8.1 J. Let f_n be the continuous piecewise linear function which is 0 on $[0,n]$, linear between the points $(n,0)$ and $(n+1,1)$ and 1 on $(0,\infty)$.

8.1 K. Given ϵ there is a δ such that $|f(x)-f(y)| < \epsilon$ whenever $|x-y| < \delta$. For all $n > \delta^{-1}$ we then have $|f(x) - f(x+\frac{1}{n})| < \epsilon$ for all x so $\|f - f_n\|_\infty < \epsilon$, establishing uniform convergence.

For a counterexample take $f(x) = x^2$. Then

$$|x^2 - (x+\frac{1}{n})^2| \geq 2x/n = 2 \text{ for } x = n,$$

so $\|f - f_n\| \geq 2$ for all n. (In fact $\|f - f_n\|_\infty = \infty$).

8.2 D. Since $f_n \to f$ uniformly we will have $\|f_n\|_\infty \to \|f\|_\infty$ and in particular $\|f_n\|_\infty$ is bounded, say by M. Then

$$\begin{aligned}\|f_n g_n - fg\|_\infty &\leq \|f_n g_n - f_n g\|_\infty + \|f_n g - fg\|_\infty \\ &\leq M\|g_n - g\|_\infty + \|g\|_\infty \|f_n - f\|_\infty \to 0.\end{aligned}$$

Note that in this argument $[a,b]$ may be replaced by any compace metric space. Pinpoint exactly where in the argument compactness (of $[a,b]$ or X) is being used.

8.3B. Sketch the graph of this function to help understand what's going on. Evidently the pointwise limit is e^{-x} for all x in $[0,\infty)$. Moreover since f_n and f agree on $[0,n]$ the infinity norm of theire difference is achieved on $[n,\infty)$. But on this last interval both functions are decreasing so

$$|f(x) - f_n(x| \leq f(x) + f_n(x) \leq f(n) + f_n(n) = 2e^{-n} \to 0.$$

This establishes uniform convergence.

For part (b) the first integral is is 1 and the integral of f_n is $1 - e^{-n} + \frac{1}{2} \to \frac{3}{2}$. This does not contradict Theorem 8.3.1 because that theorem only applies to bounded intervals, that is, not to improper integrals.

8.4C. Let $f_n(x) = x^n e^{-x}$ for $x \in [0, \infty)$. Since $xe^{-x} \le e^{-1}$ for all x we have $\|f_n\|_\infty \le e^{-n}$. Since $\sum_{n=1}^\infty e^{-n} < \infty$ the Weierstrass M-test gives uniform convergence on $[0, \infty)$.

8.4J. Note that $F(k)$ is defined since the series defining it is absolutely convergent. Define g_n on the compact metric space $X = \{1/k : k \ge 1\} \cup \{0\}$ by

$$g_n(\frac{1}{k}) = f_n(k) \text{ and } g_n(0) = \lim_{k\to\infty} f_n(k),$$

making g_n continuous at 0 hence continuous on X. We have $\|g_n\|_\infty \le M_n$ (why?) so by the M-test $\sum_{n=1}^\infty g_n$ converges uniformly to a (necessarily) continuous G. Evidently $G(\frac{1}{k}) = F(k))$ and by continuity of G we have

$$G(\frac{1}{k}) \to G(0) = \sum_{n=1}^\infty g_n(0) = \sum_{n\ge 1} L_n.$$

It is also instructive to prove this directly.

(1) Evaluate the following sums.

$$\sum_{n=1}^\infty \frac{n(n+2)}{2^n} \qquad\qquad \sum_{n=1}^\infty \frac{1}{n(n+1)5^n}$$

To do the first one start with

$$g(x) = \frac{1}{1-x} = \sum_{n\ge 0} x^n$$

and differentiate to get

$$h(x) = g'(x) = \sum_{n\ge 1} nx^{n-1}.$$

Now multiply by x^3, differentiate again and divide by x to get

$$f(x) = \frac{1}{x}\frac{d}{dx}(x^3 h(x)) = \sum_{n\ge 1} n(n+2)x^n.$$

All these operations are valid in the interval $(-1, 1)$ where the original series converges so the desired sum is $f(\frac{1}{5})$, which is left to the reader to calculate.

The second one is done similarily, by integrating the geometric series.