Math 337 Winter 2010, Problem set 5
Due Fri. Mar. 19

It is recommended that you try all the problems. Hand in only those which are starred for grading. The hints which are given are just suggestions. You may find a different or better way of doing the problem.

Please note that this problem set has two pages.

(1) *Show that if f is integrable on $[a,b]$ then $|f|$ is also and that $|\int_a^b f| \leq \int_a^b |f|$. Hint: let $f^+(x) = \max(f(x), 0)$ and $f^-(x) = \max(-f(x), 0)$ so that $|f| = f^+ + f^-$. Show that f^+ and f^- are integrable on $[a,b]$.

(2) Let V be a normed vector space. A series in V is a formal sum $\sum_{i=1}^\infty x_i$ where the x_i are elements of V. Just as in $\mathbb{R}$ the series is said to converge if the partial sums converge to a limit in V. The series is called absolutely convergent if $\sum_{i=1}^\infty \|x_i\| < \infty$. Show that if V is complete then absolute convergence implies convergence. (Imitate the proof for series of real numbers.) Derive the Weierstrass M-test (8.4.7 in DD) as a special case.

(3) *Show the converse to #2 above: if every absolutely convergent series is convergent then V is complete. Hint: suppose (x_i) is a Cauchy sequence and by passing to a subsequence assume without loss of generality that $\sum_{i=1}^\infty \|x_{i+1} - x_i\| < \infty$. Let $y_i = x_{i+1} - x_i$ and relate the partial sums of the series $\sum_{i=1}^\infty y_i$ to the sequence (x_i).

§7.1 A, B, C, D, E, J*. Also show that the monomials $f_i(x) = x^i, i = 0, 1, \ldots, n$ are linearly independent in $C[a,b]$.

(4) Use the Baire category theorem to show that an infinite dimensional Banach space cannot have a countable algebraic basis $\{x_i : i \in \mathbb{N}\}$. (This means that every finite subset of $\{x_i\}$ is linearly independent and that every $x \in V$ can be written as a linear combination of (finitely many of) the x_i's.) Hint: let E_n denote the linear span of $x_1, x_2, \ldots, x_n$, use Corollary 7.3.4. and show that E_n has no interior.

(5) Let X be a compact metric space and let $\mathrm{Lip}(X)$ denote the space of all real-valued Lipschitz functions on X, so $\mathrm{Lip}(X) \subset C(X)$. Show that $\mathrm{Lip}X$ is a closed subspace of $C(X)$. For $f \in \mathrm{Lip}(X)$ define

$$\|f\|_{\mathrm{Lip}} = \|f\|_\infty + \sup\{\frac{|f(x) - f(y|}{|x-y|} : x, y \in X, x \neq y\}.$$

Show that $\|\cdot\|_{\mathrm{Lip}}$ is a norm on $\mathrm{Lip}(X)$.

§7.2 A, C, D*, G, K

(6) §9.3 A, C, E, F*, G, K My preferred hint for F(a): for $x \in X$ define

$$O(f,x) = \lim_{r \to 0^+} \sup\{|f(x) - f(y)| : y \in B_r(x)\}$$

. (Justify the existence of the limit by showing that it is equal to $\inf_{r>0} \sup\{|f(x) - f(y)| : y \in B_r(x)\}$.) Observe that f is continuous at x if and only if $O(f,x) = 0$. Show that $E_n = \{x \in X : O(f,x) < \frac{1}{n}\}$ is open.